Feedback Nash Equilibrium and Constructions of Programmed Iteration Method

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Outline

Problem

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

- The variable u denotes a control of the first palyer.
- The variable v denotes a control of the second player.
- The first player strives to maximize $\sigma_1(x(\vartheta_0))$.
- The second player strives to maximize $\sigma_2(x(\vartheta_0))$.

Conditions

1. P and Q are compact sets.
2. f is continuous, locally lipschitzian with respect to x;
   f satisfies the sublinear growth condition with respect to x.
3. Functions σ_1, σ_2 are continuous.
4. The Isaacs condition holds.
5. The sets $\{f(t,x,u,v) : u \in P\}$ are convex for all $(t,x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $v \in Q$.
6. The sets $\{f(t,x,u,v) : v \in Q\}$ are convex for all $(t,x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $u \in P$.

If conditions 4–6 are not fulfilled, one can achieve them by using the slide regimes.

Strategies [N.N. Krasovskii, A.I. Subbotin]

Positional Strategies

- Strategy of the *first* player is $U = (u(t,x,\varepsilon), \beta_1(\varepsilon_1))$. Here $u(t,x,\varepsilon_1)$ is a function of position and precision parameter ε_1, $\beta_2(\varepsilon)$ is upper bound of fineness of partitions.
- Strategy of the *second* player is $V = (v(t,x,\varepsilon_2), \beta_2(\varepsilon_2))$.

Control formation

- The *first* player chooses partition $\Delta_1 = \{t'_j\}_{k=0}^m$, $(t'_{j+1} - t'_j) \leq \beta_1(\varepsilon_1)$. $u(t) = u(t'_j, x[t'_j], \varepsilon_1)$, $t \in [t'_j, t'_{j+1})$.
- The *second* player chooses partition $\Delta_2 = \{t''_k\}_{k=0}^l$, $(t''_{k+1} - t''_k) \leq \beta_2(\varepsilon_2)$. $v(t) = v(t''_k, x[t''_k], \varepsilon_2)$, $t \in [t''_k, t''_{k+1})$.

Bundles of Motions [Kleimenov, 1993], [N.N. Krasovskii, A.I. Subbotin]

- Step-by-step motion $x[t, t_*, x_*; U, \varepsilon_1, \Delta_1; V, \varepsilon_2, \Delta_2]$.
- The set of step-by-step motions $X(t_*, x_*; U, V, \varepsilon_1, \varepsilon_2)$.
- Consistent motion $\varepsilon_1 = \varepsilon_2$.
- The set of consistent step-by-step motions $X^c(t_*, x_*; U, V, \varepsilon)$.
- The set of ideal motions $X(t_*, x_*; U, V)$.
- The set of consistent ideal motions $X^c(t_*, x_*; U, V)$.

The function $x[\cdot] : [t_*, \vartheta_0] \to \mathbb{R}^n$, $x[t_*] = x_*$, is called *ideal motion* if there exist $\{(t_k, x_k)\}$, $\{\varepsilon_1^k\}$, $\{\varepsilon_2^k\}$, Δ_1^k, Δ_2^k such that $\mathsf{fineness}(\Delta_i^k) \leq \beta_i(\varepsilon_i^k)$ $\varepsilon_i^k \to 0$, as $k \to \infty$, and

$$x[\cdot, t_k, x_k; U, \varepsilon_1^k, \Delta_1^k; V, \varepsilon_2^k, \Delta_2^k] \rightrightarrows x[\cdot], \quad k \to \infty.$$

Feedback Nash Equilibrium [Kleimenov, 1993]

The pair of strategies (U^N, V^N) is called Nash equilibrium solution at the position (t_*, x_*) if the following inequalities hold:

$$\max\{\sigma_1(x[\vartheta_0]) : x[\cdot] \in X(t_*, x_*, U, V^N)\} \leq$$
$$\leq \min\{\sigma_1(x^c[\vartheta_0]) : x^c[\cdot] \in X^c(t_*, x_*, U^N, V^N)\}.$$

$$\max\{\sigma_2(x[\vartheta_0]) : x[\cdot] \in X(t_*, x_*, U^N, V)\} \leq$$
$$\leq \min\{\sigma_2(x^c[\vartheta_0]) : x^c[\cdot] \in X^c(t_*, x_*, U^N, V^N)\}.$$

Feedback Nash Equilibrium

The pair $(\sigma_1(x[\vartheta_0]), \sigma_2(x[\vartheta_0]))$ for $x[\cdot] \in X^c(t_*, x_*; U^N, V^N)$ is called N-value.
Denote the set of N-values at the position (t_*, x_*) by $\mathcal{N}(t_*, x_*)$.

Property [Kleimenov, 1993]: The set $\mathcal{N}(t_*, x_*)$ is nonempty. Also the set $\mathcal{N}(t_*, x_*)$ can contain more than one pair.

Auxiliary constructions

Let $\mathcal{U}$ denote the set of all measurable function from $[t_0, \vartheta_0]$ to P; let $\mathcal{V}$ denote the set of all measurable function from $[t_0, \vartheta_0]$ to Q.
Denote the solution of

$$\dot{x} = f(t, x, u(t), v(t)), \quad x(t_*) = x_*$$

by $x(\cdot, t_*, x_*, u, v)$.

We consider elements of P and Q as constant controls.

Programmed absorption operators, [Chentsov, 1975], [Chentsov, 1976]

Let $c(t,x) \in C([t_0,\vartheta_0] \times \mathbb{R}^n)$.

Definition.

$$A_1[c](t,x) \triangleq \max_{u \in P} \max_{\tau \in [t,\vartheta_0]} \min_{v \in \mathcal{V}} c(\tau, x(\tau,t,x,u,v)),$$

$$A_2[c](t,x) \triangleq \max_{v \in Q} \max_{\tau \in [t,\vartheta_0]} \min_{u \in \mathcal{U}} c(\tau, x(\tau,t,x,u,v)).$$

Auxiliary definition.
Let $(t_*,x_*) \in [t_0,\vartheta_0] \times \mathbb{R}^n$. The set of solutions of differential inclusion

$$\dot{x} \in \mathrm{co}\{f(t,x,u,v) : u \in P, v \in Q\}, \quad x(t_*) = x_*$$

denote by $\mathrm{Sol}(t_*,x_*)$.

Theorem 1

The set of N-values at position (t_*, x_*) is equal to the set $\{(c_1(t_*, x_*), c_2(t_*, x_*))\}$ where the functions $c_1(t, x)$ and $c_2(t, x)$ are continuous and satisfy the following conditions:

(C1) $c_1(\vartheta_0, \cdot) = \sigma_1(\cdot)$, $c_2(\vartheta_0, \cdot) = \sigma_2(\cdot)$;

(C2) $A_1[c_1] = c_1$; $A_2[c_2] = c_2$;

(C3) there exists $y[\cdot] \in \mathrm{Sol}(t_*, x_*)$ such that

$$c_1(t, y(t)) = c_1(t_*, x_*), \quad c_2(t, y(t)) = c_2(t_*, x_*) \ \forall t \in [t_*, \vartheta_0].$$

Proof

Sufficiency. The penalty strategies are used.
Necessity. Let $(J_1, J_2) \in \mathcal{N}(t_*, x_*)$.
Put for $i = 1, 2$

$$c_i^*(t,x) \triangleq \max_{u \in P} \min_{v \in \mathcal{V}} \sigma_1(t_*, x(t_*, t, x, u, v)),$$

$$c_i^+(t,x) \triangleq \sup_{u \in \mathcal{U}, v \in \mathcal{V}} \sigma_i(x(\vartheta_0, t, x, u, v)),$$

$$c_i^0(t,x) \triangleq \max\{c_i^*(t,x), \min\{J_i, c_i^+(t,x)\}\}, \quad i = 1, 2,$$

$$c_i^k \triangleq A_i[c_i^{k-1}], \quad i = 1, 2, \quad k \in \mathbb{N}.$$

$$c_i(t,x) \triangleq \lim_{k \to \infty} c_i^k(t,x).$$

The pair (c_1, c_2) *satisfies the condition* (C1)–(C3) *at the position* (t_*, x_*).

Equivalent Formulation of Condition (C2)

- The function c_1 is *upper minimax (viscosity) solution* of equation
$$\frac{\partial c}{\partial t} + H_1(t, x, \nabla c) = 0.$$
- The function c_2 is *upper minimax (viscosity) solution* of equation
$$\frac{\partial c}{\partial t} + H_2(t, x, \nabla c) = 0.$$

Here the function H_1 is defined by the rule
$$H_1(t, x, s) \triangleq \max_{u \in P} \min_{v \in Q} \langle s, f(t, x, u, v) \rangle,$$
the function H_2 is defined by the rule
$$H_1(t, x, s) \triangleq \max_{v \in Q} \min_{u \in P} \langle s, f(t, x, u, v) \rangle.$$

The definitions of minimax solution are given in [Subbotin, 1995].

Equivalent Formulation of Condition (C3)

There exists closed set $E \subset [t_0, \vartheta_0] \times \mathbb{R}^n$, $(t_*, x_*) \in E$, such that the graph $\mathrm{gr}(c_1, c_2; E)$ is weakly invariant under differential inclusion

$$\begin{pmatrix} \dot{x} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} \in \mathcal{F}^*(t, x) \triangleq \mathrm{co}\left\{ \begin{pmatrix} f(t, x, u, v) \\ 0 \\ 0 \end{pmatrix} : u \in P, v \in Q \right\}.$$

Here $\mathrm{gr}(c_1, c_2; E)$ denotes the graph of restriction of (c_1, c_2) on E.

$$\mathrm{gr}(c_1, c_2; E) \triangleq \{(t, x, c_1(t, x), c_2(t, x)) : (t, x) \in E\}.$$

Equivalent Formulation of Condition (C3)

There exists closed set $E \subset [t_0, \vartheta_0] \times \mathbb{R}^n$, $(t_*, x_*) \in E$, such that for all $(t, x, z_1, z_2) \in \mathrm{gr}(c_1, c_2)$

$$\Big[\mathrm{co} D_t(\mathrm{gr}(c_1, c_2; E))(t, x, z_1, z_2)\Big] \cap \mathcal{F}^*(t, x) \neq \varnothing.$$

Here $D_t(\mathrm{gr}(c_1, c_2; E))(t, x, z_1, z_2)$ is right-side derivative of set $\mathrm{gr}(c_1, c_2; E)$ for (t, x, z_1, z_2) [Guseinov, Subbotin, Ushakov]:

$$D_t(\mathrm{gr}(c_1, c_2; E))(t, x, z_1, z_2) = \Big\{(g, \zeta_1, \zeta_2) : (g, \zeta_1, \zeta_2) :$$

$$\liminf_{\delta \downarrow 0} \frac{d((x + \delta g, z_1 + \delta\zeta_1, z_2 + \delta\zeta_2), \mathrm{gr}(c_1, c_2; E; t))}{\delta} = 0\Big\},$$

$$\mathrm{gr}(c_1, c_2; E; t) \triangleq \{(x, z_1, z_2) : (t, x, z_1, z_2) \in \mathrm{gr}(c_1, c_2; E)\}.$$

System of HJ Equations [Basar & Olsder]

Let $\gamma_1, \gamma_2 : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions

$$\gamma_i(\vartheta_0, \cdot) = \sigma_i(\cdot), \quad i = 1, 2$$

$$\frac{\partial \gamma_i(t,x)}{\partial t} + \left\langle \frac{\partial \gamma_i(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),\hat{v}(t,x)) \right\rangle = 0, \quad i = 1, 2.$$

Here

$$\left\langle \frac{\partial \gamma_1(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),\hat{v}(t,x)) \right\rangle = \\ = \max_{u \in P} \left\langle \frac{\partial \gamma_1(t,x)}{\partial x}, f(t,x,u,\hat{v}(t,x)) \right\rangle,$$

$$\left\langle \frac{\partial \gamma_2(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),\hat{v}(t,x)) \right\rangle = \\ = \max_{v \in P} \left\langle \frac{\partial \gamma_2(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),v) \right\rangle.$$

Theorem 2

If functions γ_i, $i = 1, 2$ are continuously differentiable and provide the solution of system of Hamilton-Jacobi equations then for all $(t_*, x_*) \in [t_0, \vartheta_0] \times \mathbb{R}^n$ the pair of functions (γ_1, γ_2) satisfies the conditions (C1)–(C3).

Example

Differential game

$t \in [0,1]$, $u, v \in [-1,1]$

$$\begin{cases} \dot{x} & = & u \\ \dot{y} & = & v. \end{cases}$$

Functionals

$\sigma_1(x,y) \triangleq -|x-y|$, $\sigma_2(x,y) \triangleq y$.

The fixed point of operator A_1

Denote by $c_1^\diamond$ the least function satisfying the conditions $c_1(1,x,y) = \sigma_1(x_*,y_*) = -|x-y|$, $A_1[c_1] = c_1$.
The Programmed Iterations Method gives that

$$c_1^\diamond = -|x_* - y_*|.$$

Let

$$c_1^+(t,x_*,y_*) \triangleq$$

$$\triangleq \max\{\sigma_1(x[1],y[1]) : (x[\cdot],y[\cdot]) \in \mathrm{Sol}(t,x_*,y_*)\} =$$

$$= \min\{-|x_* - y_*| + 2(1-t), 0\}.$$

The functions

$$c_1^\beta(t,x_*,y_*) = -|x_* - y_*| + \beta(1-t)$$

for $\beta \in [0, c_1^+(1,x_*,y_*) - c_1^\diamond(1,x_*,y_*)]$ satisfy the conditions: $c_1^\beta(1,x,y) = \sigma_1(x,y) = -|x-y|$, $A_1[c_1^\beta] = c_1^\beta$.

The fixed point of operator A_1

Conditions $c_1(1, x_*, y_*) = \sigma_1(x_*, y_*)$ and $A_1[c_1] = c_1$ imply that $c_1(t, x_*, y_*) \in [c_1^{\diamond}(t, x_*, y_*), c_1^{+}(t, x_*, y_*)]$.

Moreover

$$\{c_1^{\beta}(t, x_*, y_*) : \beta \in [0, c_1^{+}(1, x_*, y_*) - c_1^{\diamond}(1, x_*, y_*)]\} = \\ = [c_1^{\diamond}(t, x, y), c_1^{+}(t, x, y)].$$

The fixed point of operator A_2

Only the function

$$c_2^*(t,x,y) = y_* + (1-t)$$

satisfies the conditions (C1) and (C2): $c_2^*(1,x,y) = y$, $A_2[c_2^*] = c_2^*$.

The set of N-values

Case $x_* \leq y_*$. If pair (c_1, c_2^*) satisfies the condition (C3) at the position (t, x_*, y_*) then one can directly shows that
$c_1 = -|x_* - y_*|$.
Therefore,

$$\mathcal{N}(t, x_*, y_*) = \{(-|x_* - y_*|, y_* + (1-t))\}.$$

Case $x_* > y_*$. If $\beta \in [0, c_1^+(1, x_*, y_*) - c_1^\diamond(1, x_*, y_*)]$ then the pair (c_1^β, c_2) satisfies the condition (C3) at the position (t, x_*, y_*).
Therefore,

$$\mathcal{N}(t, x_*, y_*) =$$
$$= [-|x_* - y_*|, \min\{0, -|x_* - y_*| + 2(1-t)\}] \times \{y_* + (1-t)\}.$$

The System of HJ Equations Approach

$$\begin{cases} \frac{\partial \gamma_1}{\partial t} + \frac{\partial \gamma_1}{\partial x} u_*(t,x,y) + \frac{\partial \gamma_1}{\partial y} v_*(t,x,y) & = 0 \\ \frac{\partial \gamma_2}{\partial t} + \frac{\partial \gamma_2}{\partial x} u_*(t,x,y) + \frac{\partial \gamma_2}{\partial y} v_*(t,x,y) & = 0. \end{cases}$$

Boundary conditions: $\gamma_1(1,x,y) = -|x-y|$, $\gamma_2(1,x,y) = y$.

Here $u_*(t,x,y)$ and $v_*(t,x,y)$ satisfy the conditions

$$\frac{\partial \gamma_1}{\partial x} u_*(t,x,y) = \max_{u \in P} \left[\frac{\partial \gamma_1}{\partial x} u \right], \quad \frac{\partial \gamma_1}{\partial x} v_*(t,x,y) = \max_{u \in P} \left[\frac{\partial \gamma_1}{\partial x} v \right].$$

There are no classical solution of this system.

The System of HJ Equations Approach

The minimax (viscosity in sense of Crandall and Lions) solution is unique. It is equal to

$$\gamma_1(t,x,y) = \begin{cases} x-y, & x \le y, \\ -x+y+2(1-t), & x > y, -x+y+2(1-t) < 0, \\ 0, & x > y, -x+y+2(1-t) \ge 0 \end{cases}$$

$$\gamma_2(t,x,y) = c_2^*(t,x,y) = y + (1-t).$$

Property:

$$\gamma_1(t,x,y) = \max\{J_1 : \exists J_2 (J_1, J_2) \in \mathcal{N}(t,x,y)\}.$$

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