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# Feedback Nash Equilibrium and Constructions of Programmed Iteration Method

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### Outline

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### Problem

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$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

- $\blacksquare$  The variable u denotes a control of the first palyer.
- lacksquare The variable v denotes a control of the second player.
- The first player strives to maximize  $\sigma_1(x(\vartheta_0))$ .
- The second player strives to maximize  $\sigma_2(x(\vartheta_0))$ .

### Conditions

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- $\blacksquare$  P and Q are compact sets.
- $\mathbf{2}$  f is continuous, locally lipschitzian with respect to x; f satisfies the sublinear growth condition with respect to x.
- **3** Functions  $\sigma_1$ ,  $\sigma_2$  are continuous.
- 4 The Isaacs condition holds.
- The sets  $\{f(t, x, u, v) : u \in P\}$  are convex for all  $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n, v \in Q$ .
- The sets  $\{f(t, x, u, v) : v \in Q\}$  are convex for all  $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n, u \in P$ .

If conditions 4-6 are not fulfilled, one can achieve them by using the slide regimes.

# Strategies [N.N. Krasovskii, A.I. Subbotin]

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### Positional Strategies

- Strategy of the *first* player is  $U = (u(t, x, \varepsilon), \beta_1(\varepsilon_1))$ . Here  $u(t, x, \varepsilon_1)$  is a function of position and precision parameter  $\varepsilon_1$ ,  $\beta_2(\varepsilon)$  is upper bound of fineness of partitions.
- Strategy of the second player is  $V = (v(t, x, \varepsilon_2), \beta_2(\varepsilon_2))$ .

### Control formation

- The first player chooses partition  $\Delta_1 = \{t'_j\}_{k=0}^m$ ,  $(t'_{j+1} t'_j) \leq \beta_1(\varepsilon_1)$ .  $u(t) = u(t'_j, x[t'_j], \varepsilon_1)$ ,  $t \in [t'_j, t'_{j+1})$ .
- The second player chooses partition  $\Delta_2 = \{t_k''\}_{k=0}^l$ ,  $(t_{k+1}'' t_k'') \leq \beta_2(\varepsilon_2)$ .  $v(t) = v(t_k'', x[t_k''], \varepsilon_2)$ ,  $t \in [t_k'', t_{k+1}'']$ .

# Bundles of Motions [Kleimenov, 1993], [N.N. Krasovskii, A.I. Subbotin]

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- Step-by-step motion  $x[t, t_*, x_*; U, \varepsilon_1, \Delta_1; V, \varepsilon_2, \Delta_2]$ .
- The set of step-by-step motions  $X(t_*, x_*; U, V, \varepsilon_1, \varepsilon_2)$ .
- Consistent motion  $\varepsilon_1 = \varepsilon_2$ .
- The set of consistent step-by-step motions  $X^c(t_*, x_*; U, V, \varepsilon)$ .
- The set of ideal motions  $X(t_*, x_*; U, V)$ .
- The set of consistent ideal motions  $X^c(t_*, x_*; U, V)$ .

The function  $x[\cdot]:[t_*,\vartheta_0]\to\mathbb{R}^n,\,x[t_*]=x_*,$  is called *ideal motion* if there exist  $\{(t_k,x_k)\},\,\{\varepsilon_1^k\},\,\{\varepsilon_2^k\},\,\Delta_1^k,\,\Delta_2^k$  such that fineness $(\Delta_i^k)\le\beta_i(\varepsilon_i^k)\,\,\varepsilon_i^k\to 0,$  as  $k\to\infty,$  and

$$x[\cdot, t_k, x_k; U, \varepsilon_1^k, \Delta_1^k; V, \varepsilon_2^k, \Delta_2^k] \rightrightarrows x[\cdot], \quad k \to \infty.$$

# Feedback Nash Equilibrium [Kleimenov, 1993]

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The pair of strategies  $(U^N, V^N)$  is called Nash equilibrium solution at the position  $(t_*, x_*)$  if the following inequalities hold:

$$\max\{\sigma_{1}(x[\vartheta_{0}]) : x[\cdot] \in X(t_{*}, x_{*}, U, V^{N})\} \leq \\ \leq \min\{\sigma_{1}(x^{c}[\vartheta_{0}]) : x^{c}[\cdot] \in X^{c}(t_{*}, x_{*}, U^{N}, V^{N})\}.$$

$$\max\{\sigma_{2}(x[\vartheta_{0}]) : x[\cdot] \in X(t_{*}, x_{*}, U^{N}, V)\} \leq$$

$$\leq \min\{\sigma_2(x^c[\vartheta_0]) : x^c[\cdot] \in X^c(t_*, x_*, U^N, V^N)\}.$$

# Feedback Nash Equilibrium

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The pair  $(\sigma_1(x[\vartheta_0]), \sigma_2(x[\vartheta_0]))$  for  $x[\cdot] \in X^c(t_*, x_*; U^N, V^N)$  is called N-value.

Denote the set of N-values at the position  $(t_*, x_*)$  by  $\mathcal{N}(t_*, x_*)$ .

*Property* [Kleimenov, 1993]: The set  $\mathcal{N}(t_*, x_*)$  is nonempty. Also the set  $\mathcal{N}(t_*, x_*)$  can contain more than one pair.

# Auxiliary constructions

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Let  $\mathcal{U}$  denote the set of all measurable function from  $[t_0, \vartheta_0]$  to P; let  $\mathcal{V}$  denote the set of all measurable function from  $[t_0, \vartheta_0]$  to Q. Denote the solution of

$$\dot{x} = f(t, x, u(t), v(t)), \quad x(t_*) = x_*$$

by  $x(\cdot, t_*, x_*, u, v)$ .

We consider elements of P and Q as constant controls.

# Programmed absorption operators, [Chentsov, 1975], [Chentsov, 1976]

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Let  $c(t,x) \in C([t_0,\vartheta_0] \times \mathbb{R}^n)$ .

Definition.

$$A_1[c](t,x) \triangleq \max_{u \in P} \max_{\tau \in [t,\vartheta_0]} \min_{v \in \mathcal{V}} c(\tau, x(\tau,t,x,u,v)),$$

$$A_2[c](t,x) \triangleq \max_{v \in Q} \max_{\tau \in [t,\vartheta_0]} \min_{u \in \mathcal{U}} c(\tau, x(\tau,t,x,u,v)).$$

Auxiliary definition.

Let  $(t_*, x_*) \in [t_0, \vartheta_0] \times \mathbb{R}^n$ . The set of solutions of differential inclusion

$$\dot{x} \in \text{co}\{f(t, x, u, v) : u \in P, v \in Q\}, \ x(t_*) = x_*$$

denote by  $Sol(t_*, x_*)$ .

### Theorem 1

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The set of N-values at position  $(t_*, x_*)$  is equal to the set  $\{(c_1(t_*, x_*), c_2(t_*, x_*))\}$  where the functions  $c_1(t, x)$  and  $c_2(t, x)$  are continuous and satisfy the following conditions:

(C1) 
$$c_1(\vartheta_0,\cdot) = \sigma_1(\cdot), c_2(\vartheta_0,\cdot) = \sigma_2(\cdot);$$

(C2) 
$$A_1[c_1] = c_1; A_2[c_2] = c_2;$$

(C3) there exists  $y[\cdot] \in Sol(t_*, x_*)$  such that

$$c_1(t, y(t)) = c_1(t_*, x_*), \quad c_2(t, y(t)) = c_2(t_*, x_*) \ \forall t \in [t_*, \vartheta_0].$$

### Proof

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Sufficiency. The penalty strategies are used.

Necessity. Let  $(J_1, J_2) \in \mathcal{N}(t_*, x_*)$ .

Put for i = 1, 2

$$c_i^*(t, x) \triangleq \max_{u \in P} \min_{v \in \mathcal{V}} \sigma_1(t_*, x(t_*, t, x, u, v)),$$

$$c_i^+(t,x) \triangleq \sup_{u \in \mathcal{U}, v \in \mathcal{V}} \sigma_i(x(\vartheta_0, t, x, u, v)),$$

$$c_i^0(t, x) \triangleq \max\{c_i^*(t, x), \min\{J_i, c_i^+(t, x)\}\}, \quad i = 1, 2,$$
  
$$c_i^k \triangleq A_i[c_i^{k-1}], \quad i = 1, 2, \quad k \in \mathbb{N}.$$

$$c_i(t,x) \triangleq \lim_{k \to \infty} c_i^k(t,x).$$

The pair  $(c_1, c_2)$  satisfies the condition (C1)–(C3) at the position  $(t_*, x_*)$ .

# Equivalent Formulation of Condition (C2)

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The function  $c_1$  is upper minimax (viscosity) solution of equation

$$\frac{\partial c}{\partial t} + H_1(t, x, \nabla c) = 0.$$

■ The function  $c_2$  is upper minimax (viscosity) solution of equation

$$\frac{\partial c}{\partial t} + H_2(t, x, \nabla c) = 0.$$

Here the function  $H_1$  is defined by the rule

$$H_1(t, x, s) \triangleq \max_{u \in P} \min_{v \in Q} \langle s, f(t, x, u, v) \rangle,$$

the function  $H_2$  is defined by the rule

$$H_1(t, x, s) \triangleq \max_{v \in Q} \min_{u \in P} \langle s, f(t, x, u, v) \rangle.$$

The definitions of minimax solution are given in [Subbotin, 1995].



# Equivalent Formulation of Condition (C3)

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There exists closed set  $E \subset [t_0, \vartheta_0] \times \mathbb{R}^n$ ,  $(t_*, x_*) \in E$ , such that the graph  $gr(c_1, c_2; E)$  is weakly invariant under differential inclusion

$$\begin{pmatrix} \dot{x} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} \in \mathcal{F}^*(t,x) \triangleq \operatorname{co} \left\{ \begin{pmatrix} f(t,x,u,v) \\ 0 \\ 0 \end{pmatrix} : u \in P, v \in Q \right\}.$$

Here  $gr(c_1, c_2; E)$  denotes the graph of restriction of  $(c_1, c_2)$  on E.

$$\operatorname{gr}(c_1, c_2; E) \triangleq \{(t, x, c_1(t, x), c_2(t, x)) : (t, x) \in E\}.$$

# Equivalent Formulation of Condition (C3)

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There exists closed set  $E \subset [t_0, \vartheta_0] \times \mathbb{R}^n$ ,  $(t_*, x_*) \in E$ , such that for all  $(t, x, z_1, z_2) \in \operatorname{gr}(c_1, c_2)$ 

$$\left[\operatorname{co} D_t(\operatorname{gr}(c_1, c_2; E))(t, x, z_1, z_2)\right] \cap \mathcal{F}^*(t, x) \neq \varnothing.$$

Here  $D_t(\operatorname{gr}(c_1, c_2; E))(t, x, z_1, z_2)$  is right-side derivative of set  $\operatorname{gr}(c_1, c_2; E)$  for  $(t, x, z_1, z_2)$  [Guseinov, Subbotin, Ushakov]:

$$D_{t}(\operatorname{gr}(c_{1}, c_{2}; E))(t, x, z_{1}, z_{2}) = \left\{ (g, \zeta_{1}, \zeta_{2}) : (g, \zeta_{1}, \zeta_{2}) : \lim_{\delta \downarrow 0} \frac{d((x + \delta g, z_{1} + \delta \zeta_{1}, z_{2} + \delta \zeta_{2}), \operatorname{gr}(c_{1}, c_{2}; E; t))}{\delta} = 0 \right\},$$

$$\operatorname{gr}(c_1, c_2; E; t) \triangleq \{(x, z_1, z_2) : (t, x, z_1, z_2) \in \operatorname{gr}(c_1, c_2; E)\}.$$

# System of HJ Equations [Basar & Olsder]

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$$\gamma_i(\vartheta_0,\cdot) = \sigma_i(\cdot), \quad i = 1, 2$$

$$\frac{\partial \gamma_i(t,x)}{\partial t} + \left\langle \frac{\partial \gamma_i(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),\hat{v}(t,x)) \right\rangle = 0, \quad i = 1, 2.$$

Here

$$\left\langle \frac{\partial \gamma_1(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),\hat{v}(t,x)) \right\rangle =$$

$$= \max_{u \in P} \left\langle \frac{\partial \gamma_1(t,x)}{\partial x}, f(t,x,u,\hat{v}(t,x)) \right\rangle,$$

$$\left\langle \frac{\partial \gamma_2(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),\hat{v}(t,x)) \right\rangle =$$

$$= \max_{v \in P} \left\langle \frac{\partial \gamma_2(t,x)}{\partial x}, f(t,x,\hat{u}(t,x),v) \right\rangle.$$

### Theorem 2

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System of HJ Equations

If functions  $\gamma_i$ , i = 1, 2 are continuously differentiable and provide the solution of system of Hamilton-Jacobi equations then for all  $(t_*, x_*) \in [t_0, \theta_0] \times \mathbb{R}^n$  the pair of functions  $(\gamma_1, \gamma_2)$  satisfies the conditions (C1)-(C3).

# Example

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### Differential game

$$t \in [0,1], \, u,v \in [-1,1]$$

$$\begin{cases} \dot{x} &= u \\ \dot{y} &= v \end{cases}$$

### **Functionals**

$$\sigma_1(x,y) \triangleq -|x-y|, \ \sigma_2(x,y) \triangleq y.$$

# The fixed point of operator $A_1$

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Denote by  $c_1^{\diamond}$  the least function satisfying the conditions  $c_1(1, x, y) = \sigma_1(x_*, y_*) = -|x - y|, A_1[c_1] = c_1.$ 

The Programmed Iterations Methood gives that

$$c_1^{\diamond} = -|x_* - y_*|.$$

Let

$$c_1^+(t, x_*, y_*) \triangleq \\ \triangleq \max\{\sigma_1(x[1], y[1]) : (x[\cdot], y[\cdot]) \in \operatorname{Sol}(t, x_*, y_*)\} = \\ = \min\{-|x_* - y_*| + 2(1 - t), 0\}.$$

The functions

$$c_1^{\beta}(t, x_*, y_*) = -|x_* - y_*| + \beta(1 - t)$$

for  $\beta \in [0, c_1^+(1, x_*, y_*) - c_1^{\diamond}(1, x_*, y_*)]$  satisfy the conditions:  $c_1^{\beta}(1, x, y) = \sigma_1(x, y) = -|x - y|, A_1[c_1^{\beta}] = c_1^{\beta}.$ 

# The fixed point of operator $A_1$

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Conditions  $c_1(1, x_*, y_*) = \sigma_1(x_*, y_*)$  and  $A_1[c_1] = c_1$  imply that  $c_1(t, x_*, y_*) \in [c_1^{\diamond}(t, x_*, y_*), c_1^+(t, x_*, y_*)].$ 

Moreover

$$\begin{aligned} \{c_1^{\beta}(t,x_*,y_*): \beta \in [0,c_1^+(1,x_*,y_*)-c_1^{\diamond}(1,x_*,y_*)]\} &= \\ &= [c_1^{\diamond}(t,x,y),c_1^+(t,x,y)]. \end{aligned}$$

# The fixed point of operator $A_2$

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Only the function

$$c_2^*(t, x, y) = y_* + (1 - t)$$

satisfies the conditions (C1) and (C2):  $c_2^*(1, x, y) = y$ ,  $A_2[c_2^*] = c_2^*$ .

### The set of N-values

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Case  $x_* \leq y_*$ . If pair  $(c_1, c_2^*)$  satisfies the condition (C3) at the position  $(t, x_*, y_*)$  then one can directly shows that  $c_1 = -|x_* - y_*|$ .

Therefore,

$$\mathcal{N}(t, x_*, y_*) = \{(-|x_* - y_*|, y_* + (1 - t))\}.$$

Case  $x_* > y_*$ . If  $\beta \in [0, c_1^+(1, x_*, y_*) - c_1^{\diamond}(1, x_*, y_*)]$  then the pair  $(c_1^{\beta}, c_2)$  satisfies the condition (C3) at the position  $(t, x_*, y_*)$ . Therefore,

$$\mathcal{N}(t, x_*, y_*) =$$
=  $[-|x_* - y_*|, \min\{0, -|x_* - y_*| + 2(1-t)\}] \times \{y_* + (1-t)\}.$ 

# The System of HJ Equations Approach

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Example

$$\left\{ \begin{array}{ll} \frac{\partial \gamma_1}{\partial t} + \frac{\partial \gamma_1}{\partial x} u_*(t,x,y) + \frac{\partial \gamma_1}{\partial y} v_*(t,x,y) &= 0 \\ \frac{\partial \gamma_2}{\partial t} + \frac{\partial \gamma_2}{\partial x} u_*(t,x,y) + \frac{\partial \gamma_2}{\partial y} v_*(t,x,y) &= 0. \end{array} \right.$$

Boundary conditions:  $\gamma_1(1, x, y) = -|x - y|, \ \gamma_2(1, x, y) = y.$ 

Here  $u_*(t, x, y)$  and  $v_*(t, x, y)$  satisfy the conditions

$$\frac{\partial \gamma_1}{\partial x} u_*(t,x,y) = \max_{u \in P} \left[ \frac{\partial \gamma_1}{\partial x} u \right], \quad \frac{\partial \gamma_1}{\partial x} v_*(t,x,y) = \max_{u \in P} \left[ \frac{\partial \gamma_1}{\partial x} v \right].$$

There are no classical solution of this system.

# The System of HJ Equations Approach

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The minimax (viscosity in sense of Crandall and Lions) solution is unique. It is equal to

$$\gamma_1(t, x, y) = \begin{cases} x - y, & x \le y, \\ -x + y + 2(1 - t), & x > y, -x + y + 2(1 - t) < 0, \\ 0, & x > y, -x + y + 2(1 - t) \ge 0 \end{cases}$$

$$\gamma_2(t, x, y) = c_2^*(t, x, y) = y + (1 - t).$$

### Property:

$$\gamma_1(t, x, y) = \max\{J_1 : \exists J_2(J_1, J_2) \in \mathcal{N}(t, x, y)\}.$$

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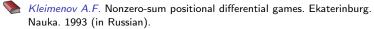
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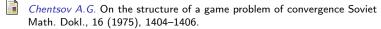
result Equivalent

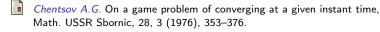
Equivalent Formulations

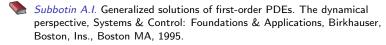
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