On Character of the Programmed Iteration Method Convergence for Control Problems with Elements of Uncertainty

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The control systems of the form:

$$\dot{x} = f(t, x, u, v),\ t \in [t_0, \vartheta_0],\ x \in \mathbb{R}^n,\ u \in P, v \in Q.$$

are considered. The variables u and v are called the controls of first and second players, respectively.

Purposes

Let $M \subset [t_0, \vartheta_0] \times \mathbb{R}^n$.
The first player is trying to bring the system onto the set M.
The second player is trying to prevent this meeting.

Condition

- M is a close set.
- P and Q are finite-dimensional compacts.
- the function $f : [t_0, \vartheta_0] \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ is continuous.
- the function f is locally Lipschitz with respect to x.
- f satisfies the sublinear growth condition with respect to x.
- for any $t \in [t_0, \vartheta_0]$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}^n$,

  $$\min_{u \in P} \max_{v \in Q} < s, f(t, x, u, v) >= \max_{v \in Q} \min_{u \in P} < s, f(t, x, u, v) > .$$

Consider a position (t_*, x_*).

The positional (feedback) control is used.

Let $U : [t_0, \vartheta_0] \times \mathbb{R}^n \to P$ be a positional strategy. Choose a partition $\Delta = t_* = \tau_0 < \tau_1 < \ldots < \tau_n = \vartheta_0$. Any solution of the inclusions

$$\dot{x}(t) \in \{f(t, x(t), U(\tau_i, x(\tau_i)), v) : v \in Q\}, t \in [\tau_i, \tau_{i+1}]$$

$x(t_*) = x_*$ is called step-by-step motion.

Constructive motions

Limits of these step-by-step motions is called constructive motions generated by a positional strategy U and emerging from the position (t_*, x_*).

This formalization was suggested by N. N. Krasovskii.

Positional Absorption Set

The structure of differential games solution is given by **Krasovskii-Subbotin alternative theorem**. The set of successful solvability of approach problem is called *positional absorption set*. The positional absorption set is the maximal u-stable bridge by the alternative theorem.

Definition

The set $W \subset [t_0, \vartheta_0] \times \mathbb{R}^n$ is called u-stable bridge if the following conditions holds:

1. $M \subset W$
2. $\forall v_* \in Q, \forall (t_*, x_*) \in W \exists y(\cdot)$

$$\dot{y}(t) \in \mathrm{co}\{f(t, x, u, v_*) : u \in P\},\ \ y(t_*) = x_*,$$
$$\exists \theta \in [t_*, \vartheta_0] :\ \ ((\theta, y(\theta) \in M)\&$$
$$((t, y(t)) \in W,\ \ \forall t \in [t_*, \theta]).$$

Let $E \subset [t_0, \vartheta_0] \times \mathbb{R}^n$.

$A(E) \triangleq \{(t,x) \in E|$ for all controls $v(t)$ there exists a solution of differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(t, y(t), u, v(t))|u \in P\},$$

$y(t) = x$ such that $(\theta, y(\theta)) \in M$ for some $\theta \in [t, \vartheta_0]$ and $(t, y(t)) \in E\ \forall t \in [t, \theta]\ \}$

$\mathbf{A}(E) \triangleq \{(t,x) \in E|$ for all *constant* control v^* there exists a solution of differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(t, y(t), u, v^*)|u \in P\},$$

$y(t) = x$ such that $(\theta, y(\theta)) \in M$ for some $\theta \in [t, \vartheta_0]$ and $(t, y(t)) \in E\ \forall t \in [t, \theta]\ \}$

The sequences of sets

Definition

$$W^{(0)} \triangleq [t_0, \vartheta] \times \mathbb{R}^n,$$

$$W^{(k)} = A(W^{(k-1)}),\ k > 0.$$

$$W_0 \triangleq [t_0, \vartheta] \times \mathbb{R}^n,$$

$$W_k = \mathbf{A}(W_{k-1}),\ k > 0.$$

Properties

$$W^{(k)} \downarrow \mathfrak{W}.$$

$$W_k \downarrow \mathfrak{W}.$$

$\mathfrak{W}$ – Positional Absorption set.

$$W^{(k)} \subset W_k.$$

Let M be a compact set. Then

1. Sequences $W^{(k)}$ and W_k convergence to $\mathfrak{W}$ in Hausdorff metric.
2. Either $\mathfrak{W}[t] = \varnothing$ and there exists K, such that $W^{(k)}[t] = \varnothing$, $W_k[t] = \varnothing$ for any $k > K$, or $W^{(k)}[t] \neq \varnothing$, $W_k[t] \neq \varnothing$ for any natural k and $\mathfrak{W}[t] \neq \varnothing$.
3. Let $t \in [t_0, \vartheta_0]$ be a moment such that $\mathfrak{W}[t] \neq \varnothing$. In this case the Hausdorff convergence of $W^{(k)}[t]$ and $W_k[t]$ to $\mathfrak{W}[t]$ takes place.

$$E[t] \triangleq \{x \in \mathbb{R}^n | (t, x) \in E\}.$$

Extremal Shift to an Unstable Set (Analog of Krasovskii-Subbotin rule)

Let (t_*, x_*) be a position, $\Delta = \{\tau_i\}_{i=0}^N$ be a partition of the segment $[t_*, \vartheta_0]$.

Formation of control by first player

Let x_i be a location of the system at the moment τ_i, and let $y_i^{(k)}$ be a closest element of $W^{(k)}[\tau_i]$ to the x_i. The control $u_i^{(k)}$ is defined by the rule:

$$\max_{v \in Q} < y_i^{(k)} - x_i, f(\tau_i, x_i, u_i^{(k)}, v) >= \\ = \min_{u \in P} \max_{v \in Q} < y_i^{(k)} - x_i, f(\tau_i, x_i, u, v) > .$$

Motions

The motion on $[\tau_i, \tau_{i+1}]$ is defined as a solution of equation:

$$x(t) = x_i + \int_{\tau_i}^{t} f(\xi, x(t), u_i^{(k)}, v[\xi]) d\xi.$$

Function $v[\cdot]$ is a second player control.

Extremal Shift to an Unstable Set

Theorem

Let $\tau_* \in I_0$ *be a moment such that* $\mathfrak{W}[\tau_*] \neq \varnothing$, *and let* $\varepsilon > 0$. *Then there exist* $\delta > 0$ *such that for any partition* $\Delta = \{\tau_i\}_{i=0}^N$ *of segment* $[\tau_*, \vartheta_0]$, *satisfying the condition*

$$\max_{i=\overline{0,N-1}} (\tau_{i+1} - \tau_i) \leq \delta,$$

one can choose $J \in \mathcal{N}$ *with property for all* $j > J$ *and* $x_* \in W^{(j)}[\tau_*]$

$$\exists \theta \in [\tau_*, \vartheta_0] : \; d[x[\theta], M[\theta]] \leq \varepsilon.$$

Here $x[\cdot]$ *is a motion defined by a extremal to the set* $W^{(j)}$ *shift rule.*

$d(x, A)$ is a distance between x and set $S \subset \mathbb{R}^n$.

Questions?