A Transformation of Game-Theoretical Control Problems

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International Conference
on Differential Equations and Dynamical Systems
Suzdal, June 27 - July 2, 2008

Conflict-controlled system

$$\dot{x} = f(x, u, v), \quad (1)$$
$$t \in [0, \vartheta],\ x \in \mathbb{R}^n,\ u \in P,\ v \in Q.$$

Here $u \in P$ and $v \in Q$ are the controls of the first player and the second player respectively.

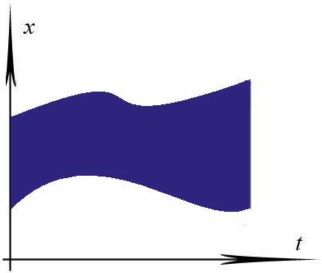
Target Set

$$M \subset [t_0, \vartheta_0] \times \mathbb{R}^n.$$

M is closed.
$F = M[\vartheta] = \{x : (\vartheta, x) \in M\}$.

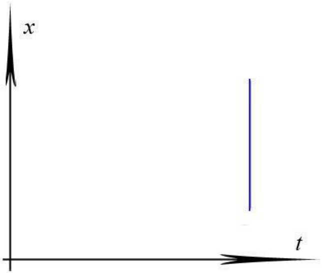
Purpose

Original Target Set



⇒

Transformed Target Set



Conditions

- P and Q are compacts in finitely dimensional spaces.
- f is continuous;
- f is locally lipschitzian with respect to x;
- f satisfies the sublinear growth condition with respect to x.

We consider differential games in the class of *counter-strategies of the first player* and *feedback strategies of the second player* (advantage of the first player).

Formalization introduced by N.N. Krasovskii and A.I. Subbotin

Step-by-step motions

Let $U : [0,\vartheta] \times \mathbb{R}^n \times Q \to P$ be a counter-strategy. Let (t_*, x_*) be a position, let $\Delta = \{\tau_i\}_{i=0}^N$ be a partition of $[t_*, \vartheta]$, let $v[\cdot]$ be a measurable of the second player. Then step-by-step motion is a solution of following equations:

$$x[t] = x[\tau_{i-1}] + \int_{\tau_{i-1}}^{t} f(x[\xi], U(\tau_{i-1}, x[\tau_{i-1}], v[\xi]), v[\xi]) d\xi,$$

$$t \in [\tau_{i-1}, \tau_i],\ x[\tau_0] = x_*.$$

The limits of step-by-step motions as fineness of partition tends to 0 are called constructive motions in sense of N.N. Krasovskii and A.I. Subbotin.

Structure of solution

By Krasovskii-Subbotin alternative theorem:

- The set of solvability of approach problem is u-stable bridge;
- The counter-strategy extreme to the maximal u-stable bridge is optimal.

Definition

A set $W \subset [0,\vartheta] \times \mathbb{R}^n$ is called u-stable bridge if $\forall (t_*, x_*) \in W \ \forall v_* \in Q \ \exists y(\cdot)$ such that

$$\dot{y}(t) \in \mathrm{co}\{f(y(t), u, v_*) : u \in P\}, \ y(t_*) = x_*, \ \exists \theta \in [t_*, \vartheta] : \\ ((\theta, y(\theta)) \in M) \& \ ((t, y(t)) \in W \ \forall t \in [t_*, \theta]).$$

Counter-strategy extreme to the set W

Let (t_*, x_*) be a position, $v_* \in Q$. Let w_* be a proximal to x_* element of $W[t_*]$.

$$U(t_*, x_*, v_*) \triangleq \mathrm{argmin}\{\langle w_* - x_*, f(x_*, u, v_*)\rangle : u \in P\}.$$

Here M is target set,

$$E[t] \triangleq \{x \in \mathbb{R}^n : (t, x) \in E\}.$$

Isaacs condition

For all $x, s \in \mathbb{R}^n$

$$\min_{u \in P} \max_{v \in Q} \langle s, f(x,u,v) \rangle = \max_{v \in Q} \min_{u \in P} \langle s, f(x,u,v) \rangle.$$

If Isaacs condition holds then there exists optimal feedback strategy of the first player $U(t,x)$. This strategy is extreme to the maximal u-stable bridge.

Original problem

Conflict controlled system

$$\dot{x} = f(x, u, v), \quad (1)$$
$$t \in [0, \vartheta],\ x \in \mathbb{R}^n,\ u \in P,\ v \in Q.$$

Variable u is a control of the first player, variable v is a control of the second player.

Target Set

Suppose that M is controllability set of the control system $g(x, \omega)$, $\omega \in \Omega$, and the target set $M^* \triangleq \{\vartheta\} \times F$:

$$M = \{(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n : \exists x_* \in F\ \ \exists \text{ measure } \mu : \\ x = \varphi_g(t, \vartheta, x_*, \mu)\}.$$

- $F = M[\vartheta]$,
- $\varphi_g(\cdot, \vartheta, x_*, \mu)$ is a motion of controlled system $\dot{x} = g(x, \omega)$, $\omega \in \Omega$, generated by measure μ.

Transformed problem

Conflict controlled system

$$\dot{x} = f^*(x, \nu, u, \omega, v),$$

$$x \in \mathbb{R}^n,\ \nu \in \{0, 1\},\ u \in P,\ \omega \in \Omega,\ v \in Q.$$

Variables u, ν and ω are controls of the first player, variable v is a control of the second player.

$$f^*(x, \nu, u, \omega, v) = \nu \cdot f(x, u, v) + (1 - \nu) \cdot g(x, \omega) = \\ = \begin{cases} f(x, u, v), & \nu = 1, \\ g(x, \omega), & \nu = 0. \end{cases}$$

Target set

$M^* \triangleq \{\vartheta\} \times F$

Programmed iteration method (A.G. Chentsov)

Program absorption operator

$A(E)$ is

the set of positions $(t_*, x_*) \in E$ for whose under any control of the second player there exists measure-control bringing the system on target set M within the set E.

Sequence of sets

$$W_0 = [0, \vartheta] \times \mathbb{R}^n;$$

$$W_k = A(W_{k-1}),\ k \in \mathbb{N}.$$

$$\mathfrak{W} = \bigcap_{k \in \mathbb{N}} W_k.$$

$\mathfrak{W}$ is the set of solvability of approach problem.

Designation

Original system

The set of solvability of approach problem is denoted by $\mathfrak{W}$.

Operator of program absorption is denoted by A.

$$W_k \triangleq A^k([t_0, \vartheta_0] \times \mathbb{R}^n), \quad k \in \mathbb{N} \cup \{0\}.$$

Transformed system

The set of solvability of approach problem is denoted by $\mathfrak{W}^*$.

Operator of program absorption is denoted by A^*.

$$W_k^* \triangleq (A^*)^k([t_0, \vartheta_0] \times \mathbb{R}^n), \quad k \in \mathbb{N} \cup \{0\}.$$

Designation

$\mathcal{F}^{\tau}_{u,v}$ is a flow for time τ generated by the constant controls $u \in P$ and $v \in Q$ in the system

$$\dot{x} = f(x, u, v).$$

$\mathcal{G}^{\tau}_{\omega}$ is a flow for time τ generated by the constant control $\omega \in \Omega$ in the system

$$\dot{x} = g(x, \omega).$$

Theorem

Assumption

For all $u \in P$, $v \in Q$, $\omega \in \Omega$ и $\tau', \tau'' \geq 0$ flows $\mathcal{F}^{\tau'}_{u,v}$ and $\mathcal{G}^{\tau''}_{\omega}$ commute:

$$\mathcal{F}^{\tau'}_{u,v} \circ \mathcal{G}^{\tau''}_{\omega} = \mathcal{G}^{\tau''}_{\omega} \circ \mathcal{F}^{\tau'}_{u,v}.$$

Statements

1. $W_k = W^*_k \ \forall k \in \mathbb{N}$;
2. $\mathfrak{W} = \mathfrak{W}^*$;
3. if original system satisfies Isaacs condition then the transformed system inherits this property.

Assumption

For all u, v and ω $f(\cdot, u, v)$ and $g(\cdot, \omega)$ are smooth functions.

Property

Flows $\mathcal{F}_{u,v}^{\tau'}$ and $\mathcal{G}_{\omega}^{\tau''}$ commute iff

$$[f(\cdot, u, v), g(\cdot, \omega)](x) = 0$$
$$\forall x \in \mathbb{R}^n \ \ \forall u \in P \ \ \forall v \in Q \ \ \forall \omega \in \Omega.$$

Here

$$[V_1, V_2](x) = \frac{\partial V_2(x)}{\partial x} V_1(x) - \frac{\partial V_1(x)}{\partial x} V_2(x).$$

Cylindric target set

Original system

$$\dot{x} = f(x,u,v), \quad t \in [0,\vartheta], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

$M = [0,\vartheta] \times F$, $F \subset \mathbb{R}^n$.

$$g(x,\omega) \equiv 0, \quad \Omega = \{\omega\}.$$

$$[f(\cdot,u,v),0] \equiv 0$$

Transformed system

$\dot{x} = u_0 f(x,u,v), \; t \in [0,\vartheta], \; x \in \mathbb{R}^n, \; u_0 \in \{0,1\}, \; u \in P, \; v \in Q.$

$M = \{\vartheta\} \times F$, $F \subset \mathbb{R}^n$.

Statement 2 of **Theorem** for this case is obtained in *Mitchel I.M., Bayen A.M., Tomlin C.J.* // IEEE Trans.Aut. Control, 2005, **50**, 7.

Pointing at sinking island

Original system

$$\begin{cases} \dot{y} = & z \\ \dot{z} = & h(u,v). \end{cases}$$

$y, z \in \mathbb{R}^m$, $u \in P$, $v \in Q$, $\vartheta = 1$.

$$M = \{(t,y,z) : t \in [0,1], \|y\| \leq 1 - t,\ \ z = 0\}.$$

$\Omega = \{\omega \in \mathbb{R}^m : \|\omega\| \leq 1\}$,
$g(x,\omega) = g(\omega) = \omega$.

$$[f(y,z,u,v), g(\omega)] =$$
$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} z \\ h(u,v) \end{pmatrix} - \begin{pmatrix} \mathbf{0} & E \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} g(\omega) \\ 0 \end{pmatrix} = 0.$$

Pointing at sinking island

Original system

$$\begin{cases} \dot{y} = & z \\ \dot{z} = & h(u,v). \end{cases}$$

$y, z \in \mathbb{R}^m$, $u \in P$, $v \in Q$, $\vartheta = 1$.

$$M = \{(t,y,z) : t \in [0,1], \|y\| \leq 1-t,\ z = 0\}.$$

Transformed system

$$\begin{cases} \dot{y} = & \nu \cdot z + (1-\nu)\cdot g(\omega) \\ \dot{z} = & \nu \cdot h(u,v). \end{cases}$$

$y, z \in \mathbb{R}^m$, $\nu \in \{0,1\}$, $u \in P$, $\omega \in \Omega$, $v \in Q$, $\vartheta = 1$.

$$M^* = \{(t,y,z) : t = 1,\quad y = z = 0\}.$$

$\Omega = \{\omega \in \mathbb{R}^m : \|\omega\| \leq 1\}$.

Questions?