Approximative Realization of Nonanticipative Strategies in Nonzero-sum Differential Games

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Outline

Nonzero-sum differential game

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

Here u and v are controls of the player I and the player II respectively.

- The player I wants to maximize $\sigma_1(x(\vartheta_0))$.
- The player II wants to maximize $\sigma_2(x(\vartheta_0))$.

Conditions

- The sets P and Q are compacts.
- The functions f, σ_1 and σ_2 are continuous.
- The function f is locally Lipschitz continuous with respect to the phase variable.
- The function f satisfies the sublinear growth condition with respect to x.
- Isaacs condition holds.

Approaches

- System of Hamilton-Jacobi equations;
- Punishment strategies.

Measure-Valued Controls

Let E be a control space. Denote by $\mathcal{R}(\tau;E)$ the set of measure-valued controls μ on $[\tau,\vartheta_0]$ with values in $\mathrm{rpm}(E)$.

- *Control of the Player I:* $\mu\in\mathcal{R}(\tau;P)$;
- *Control of the Player II:* $\nu\in\mathcal{R}(\tau;Q)$;
- *Joint control of the players:* $\eta\in\mathcal{R}(\tau;P\times Q)$.

Motion:

Denote by $x(\cdot,t_*,x_*,\eta)$ the solution of the problem

$$\dot{x}=\int_{P\times Q}f(t,x,u,v)\eta(t,d(u,v)),\quad x(t_*)=x_*.$$

Nonanticipative strategies

A map $\alpha : \mathcal{R}(t_*; Q) \to \mathcal{R}(t_*; P \times Q)$ is *nonanticipative strategy of the Player I* if

- $\alpha[\nu](t; P \times \Upsilon) = \nu(t; \Upsilon) \ \ \forall \Upsilon \subset Q$;
- $\nu_1(t, \cdot) = \nu_2(t, \cdot)$ for almost every $t \in [t_*, \theta]$ *implies* $\alpha[\nu_1](t, \cdot) = \alpha[\nu_2](t, \cdot)$ for almost every $t \in [t_*, \theta]$.

$$\mathcal{M}^1[\alpha](t_*, x_*) = \{x(\cdot, t_*, x_*, \alpha[\nu]) : \nu \in \mathcal{R}(t_*; Q)\}.$$

Nonanticipative strategies

A map $\beta : \mathcal{R}(t_*; P) \to \mathcal{R}(t_*; P \times Q)$ is *nonanticipative strategy of the Player II* if

- $\beta[\mu](t; \Upsilon \times Q) = \mu(t; \Upsilon) \ \ \forall \Upsilon \subset P$;
- $\mu_1(t, \cdot) = \mu_2(t, \cdot)$ for almost every $t \in [t_*, \theta]$ *implies* $\beta[\mu_1](t, \cdot) = \beta[\mu_2](t, \cdot)$ for almost every $t \in [t_*, \theta]$.

$$\mathcal{M}^2[\beta](t_*, x_*) = \{x(\cdot, t_*, x_*, \beta[\mu]) : \mu \in \mathcal{R}(t_*; P)\}.$$

Nash equilibrium

A pair of nonanticipative strategies α^n, β^n and the set of motions $\mathcal{S} \subset \mathcal{M}^1[\alpha^n](t_*, x_*) \cap \mathcal{M}^2[\beta^n](t_*, x_*)$ is a *Nash equilibrium* at the position (t_*, x_*) if the following inequalities hold:

$$\sup\{\sigma_1(x(\vartheta_0)) : x(\cdot) \in \mathcal{M}^2[\beta^n](t_*, x_*)\} \leq \inf\{\sigma_1(z(\vartheta_0)) : z(\cdot) \in \mathcal{S}\},$$

$$\sup\{\sigma_2(x(\vartheta_0)) : x(\cdot) \in \mathcal{M}^1[\alpha^n](t_*, x_*)\} \leq \inf\{\sigma_2(z(\vartheta_0)) : z(\cdot) \in \mathcal{S}\}.$$

Auxiliary zero-sum games

- *Game* Γ_1*:* The player I wants to maximize $\sigma_1(x(\vartheta_0))$, the purpose of the player II is opposite. Denote the value of this game by $\omega_1 : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$.

- *Game* Γ_2*:* The player II wants to maximize $\sigma_2(x(\vartheta_0))$, the purpose of the player I is opposite. Denote the value of this game by $\omega_2 : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$.

Auxiliary differential inclusion

$$\dot{x} \in \mathcal{F}(t,x) \triangleq \mathrm{co}\{f(t,x,u,v) : u \in P, v \in Q\}$$

By $\mathrm{Sol}(t_*, x_*)$ denote the set of its solution with initial data $x(t_*) = x_*$.

$$\mathrm{Sol}(t_*, x_*) = \{x(\cdot, t_*, x_*, \eta) : \eta \in \mathcal{R}(t_*; P \times Q)\}.$$

Structure of Nash equilibriums

Theorem

- If $y(\cdot) \in \mathcal{S}$ then

$$\omega_i(t, y(t)) \leq \sigma_i(y(\vartheta_0)), \quad t \in [t_*, \vartheta_0], \quad i = 1, 2. \qquad (*)$$

- If $y(\cdot) \in \mathrm{Sol}(t_*, x_*)$ satisfies condition $(*)$ then there exists the Nash equilibrium $(\alpha^n, \beta^n, \mathcal{S})$ such that

$$\{y(\cdot)\} = \mathcal{S} \subset \mathcal{M}^1[\alpha^n](t_*, x_*) \cap \mathcal{M}^2[\beta^n](t_*, x_*).$$

- There exists at least one Nash equilibrium.

The same structure has the set of Nash equilibriums in discontinuous feedback formalization.

Design of Nash equilibrium

$$y(\cdot) = x(\cdot, t_*, x_*, \eta_*).$$

Let μ_* be a projection of η_* on P: $\mu_*(t;\Gamma) \triangleq \eta_*(t, \Gamma \times Q)$.
Let ν_* be a projection of η_* on Q: $\nu_*(t;\Upsilon) \triangleq \eta_*(t, P \times \Upsilon)$.

Nonanticipative Strategy α^n: deviation from ν_* leads to punishment.

Nonanticipative Strategy β^n: deviation from μ_* leads to punishment.

Nash value of the game

$$\mathcal{N}(t_*, x_*) = \Big\{(\sigma_1(y(\vartheta_0)), \sigma_2(y(\vartheta_0))) : \\ y(\cdot) \in \mathcal{S}, (\alpha^n, \beta^n, \mathcal{S}) \text{ is a Nash equilibrium at } (t_*, x_*)\Big\}.$$

Control with Guide. Player I

x – state of the system, w – state of the guide.

Strategy of the Player I:

$$U_{mod} = (u(t,x,w,\varepsilon), \psi_1(t^+,t,x,w,\varepsilon), \chi_1(t,x,\varepsilon), \beta_1(\varepsilon)).$$

- $u(t,x,w,\varepsilon)$ forms the control;
- $\psi_1(t^+,t,x,w,\varepsilon)$ is transitional function of the guide;
- $\chi_1(t,x,\varepsilon)$ initializes the guide;
- $\beta_1(\varepsilon)$ bounds the interval between corrections of control.

Control with Guide. Player I

Let $(t^\natural, x^\natural)$ be an initial position, ε be a precision parameter, control correct at the time instants $t_0 < t_1 \leq \ldots \leq t_m$; $\Delta = \{t_j\}$; $d(\Delta) \leq \beta_1(\varepsilon)$.

Control formation.

If

at t_i the state of the system is x_i, the state of the guide is w_i,

then

on $[t_i, t_{i+1})$ the control of the Player I is $u(t_i, x_i, w_i, \varepsilon)$,

at t_{i+1} the state of the guide is $\psi_1(t_{i+1}, t_i, x_i, w_i, \varepsilon)$.

Step-by-step motion: $x^1[\cdot, t^\natural, x^\natural, U_{mod}, \varepsilon, \Delta, v[\cdot]]$.

Control with Guide

Strategy of the Player II:
$V_{mod} = (v(t, x, w, \varepsilon), \psi_2(t^+, t, x, w, \varepsilon), \chi_2(t_0, x_0), \beta_2(\varepsilon))$.

Step-by-step Motion: $x^2[\cdot, t^\natural, x^\natural, V_{mod}, \varepsilon, \Xi, u[\cdot]]$.

Consistent Motion. Precision parameters of the Players are equal.
$x^c[\cdot, t^\natural, x^\natural, U_{mod}, V_{mod}, \varepsilon, \Delta, \Xi]$.

Constructive Motions

$(t^\natural, x^\natural) \to (t_*, x_*)$, $\varepsilon \to 0$.

- *Limit Motions of the Player I:* $X^1(t_*, x_*, U_{mod})$;
- *Limit Motions of the Player II:* $X^2(t_*, x_*, V_{mod})$;
- *Limit Consistent Motions:* $X^c(t_*, x_*, U_{mod})$.

Nash equilibrium

Pair of strategies (U^n_{mod}, V^n_{mod}) with guide is the Hash equilibrium if

$$\max\{\sigma_1(x(\vartheta_0)) : x(\cdot) \in X^2(t_*, x_*, V^n_{mod})\} \\ \leq \min\{\sigma_1(z(\vartheta_0)) : z(\cdot) \in X^c(t_*, x_*, U^n_{mod}, V^n_{mod})\},$$

$$\max\{\sigma_2(x(\vartheta_0)) : x(\cdot) \in X^1(t_*, x_*, V^n_{mod})\} \\ \leq \min\{\sigma_2(z(\vartheta_0)) : z(\cdot) \in X^c(t_*, x_*, U^n_{mod}, V^n_{mod})\}.$$

Deviation

Let $\mathcal{Y}$ and $\mathcal{Z}$ be a set of continuous functions from $[t_*, \vartheta_0]$ to $\mathbb{R}^n$.

$$h(t_*, \mathcal{Y}, \mathcal{Z}) \triangleq \sup_{y(\cdot)\in\mathcal{Y}} \inf_{z(\cdot)\in\mathcal{Z}} \max_{t\in[t_*,\vartheta_0]} \|y(t) - z(t)\|.$$

Approximative realization

Theorem

Let $(\alpha^n, \beta^n, \mathcal{S})$ is a Nash equilibrium in the class of nonanticipative strategies.
There exists Nash equilibrium in the class of controls with guide (U^n_{mod}, V^*_{mod}) such that for all $t \in [t_*, \vartheta_0]$ the following equilities are fulfilled

1. $h(t_*, X^c(t_*, x_*, U_{mod}, V_{mod}), \mathcal{S}) = 0;$
2. $h(t_*, X^1(t_*, x_*, U_{mod}), \mathcal{M}^1[\alpha](t_*, x_*)) = 0;$
3. $h(t_*, X^2(t_*, x_*, V_{mod}), \mathcal{M}^2[\beta](t_*, x_*)) = 0.$

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