On a Strucure of of the Set of Differential Games Values

Yurii Averboukh

Institute of Mathematics and Mechanics UrB RAS,
Yekaterinburg, Russia
ayv@imm.uran.ru

Technion, September 6, 2010

Outline

Differential game

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

Problem of Degree.

$$\gamma(x(\cdot)) = \sigma(x(\vartheta_0)) + \int_t^{\vartheta_0} g(t, x, u, v)dt \to \min_u \max_v .$$

Problem of Guidance.
$M \subset [t_0, \vartheta_0] \times \mathbb{R}^n$.
The Player U wants to bring x to M.
The Player V prevents him.

Conditions

- f, g and σ are continuous;
- f and g satisfy the sublinear growth condition with respect to x, also they are locally Lipschitz continuous with respect to the phase variable;
- P and Q are compact sets;
- M is close.
- Isaacs condition:

$$\min_{u\in P}\max_{v\in Q}[\langle s, f(t,x,u,v)\rangle + g(t,x,u,v)] = \\ = \max_{v\in Q}\min_{u\in P}[\langle s, f(t,x,u,v)\rangle + g(t,x,u,v)].$$

Feedback formalization [Krasovskii & Subbotin]

Let $U : [t_0, \vartheta_0] \times \mathbb{R}^n \to P$ be a control of the Player U,
(t_*, x_*) be an initial point,
$\Delta = \{t_* = t_0 < t_1 < \ldots < t_m = \vartheta_0\}$ be a partition of $[t_*, \vartheta_0]$,
$v[\cdot]$ be a measurable control of the Player V.

We do not put any condition on U!

Step-by-step motions

The solution of the problem

$$\dot{x} = f(t, x[t], u[t], v[t]), \quad x[t_*] = x_*,$$

is called *step-by-step motion* if $u[t] = U(t_i, x[t_i])$ for $t \in [t_i, t_{i+1}]$.

Denote it by $x^1[\cdot, t_*, x_*, U, \Delta, v[\cdot]]$.

Strategies of the Player V

The strategy of the Player V is an arbitrary function $V : [t_0, \vartheta_0] \times \mathbb{R}^n \to Q$.
We also can construct *step-by-step motions* $x^2[\cdot, t_*, x_*, V, \Delta, u[\cdot]]$.

Approximate Guidance

Denote by M_ε the ε-neighborhood of M.

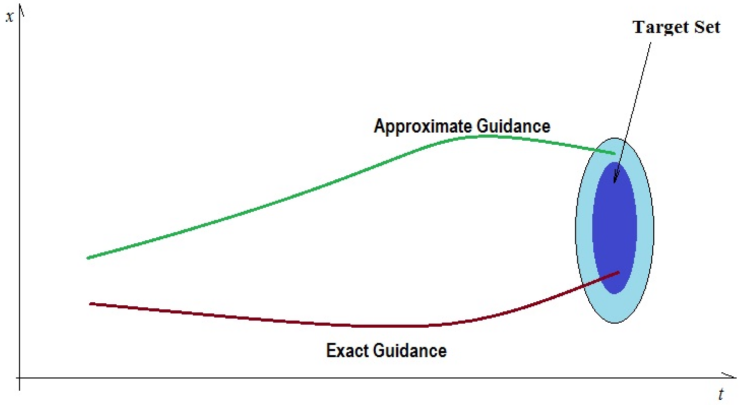
Let (t_*, x_*) be a position, find the strategy U^* with the property:

for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial position (t^*, x^*) δ-close to (t_*, x_*), any partition of the segment $[t^*, \vartheta_0]$ Δ with fineness less then δ, any control of the player V $v[\cdot]$

$$(\tau, x^1[\tau, t^*, x^*, U^*, v[\cdot]]) \in M_\varepsilon$$

for some $\tau \in [t^*, \vartheta_0]$.

Approximate Guidance



Approximate Evasion

Let (t_*, x_*) *be a position, find the strategy* V^* *with the property:*

for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial position (t^*, x^*) δ-close to (t_*, x_*), any partition of the segment $[t^*, \vartheta_0]$ Δ with fineness less then δ, any control of the player U $u[\cdot]$

$$(\tau, x^2[\tau, t^*, x^*, U^*, v[\cdot]]) \notin M_\varepsilon$$

for any $\tau \in [t^*, \vartheta_0]$.

Problem of Guidance. Alternative Theorem [Krasovskii & Subbotin]

There exists a set $W \subset [t_0, \vartheta_0] \times \mathbb{R}^n$ with the property

- if $(t_*, x_*) \in W$ then the Problem of Approximate Guidance is solvable;
- if $(t_*, x_*) \notin W$ the the Problem of Approximate Evasion is solvable.

W is maximal u-stable bridge.

u-stable bridges

The set W is called u-stable bridge if for any $(t_*, x_*) \in W$, any $v \in Q$ there exists a solution of inclusion

$$\dot{y} \in \mathrm{co}\{f(t, y, u, v) : u \in P\}, \quad y(t_*) = x_*$$

and $\tau \in [t_*, \vartheta_0]$ such that $(\tau, y(\tau)) \in M$ and for all $\xi \in [t_*, \tau]$ $(\xi, y(\xi)) \in W$.

Optimal control. Extremal shift

Let $(t_*, x_*) \in W$, denote by w_* the nearest element of the section of W:

$$\min\{\|w - x_*\| : (t_*, w) \in W\} = \|x_* - w_*\|.$$

Let $u^e \in P$ satisfy the following condition:

$$\min_{u \in P} \max_{v \in Q} \langle x_* - w_*, f(t, x, u, v)\rangle = \max_{v \in Q} \langle x_* - w_*, f(t, x, u^e, v)\rangle.$$

Put $U^e(t_*, x_*) = u^e$.

u-stable bridges

- Programmed Iteration Method (A.G. Chentsov);
- Numerical Methods (V.S. Patsko et al, V.N. Ushakov et al).

Problem of Degree

Lower Value

$$\Gamma_*(t_*, x_*, U) \triangleq \limsup_{d(\Delta)\downarrow 0, (t^*, x^*) \to (t_*, x_*)} \sup_{v[\cdot]} \gamma(x^1[\cdot, t^*, x^*, U, \Delta, v[\cdot]]).$$

$$\mathrm{Val}_*(t_*, x_*) \triangleq \inf_U \Gamma_*(t_*, x_*, U).$$

Upper Value

$$\Gamma^*(t_*, x_*, U) \triangleq \liminf_{d(\Delta)\downarrow 0, (t^*, x^*) \to (t_*, x_*)} \inf_{v[\cdot]} \gamma(x^2[\cdot, t^*, x^*, V, \Delta, u[\cdot]]).$$

$$\mathrm{Val}^*(t_*, x_*) \triangleq \sup_V \Gamma^*(t_*, x_*, V).$$

Problem of Degree

Alternative theorem states that there exists the value of the game

$$\mathrm{Val}(t_*, x_*) = \mathrm{Val}_*(t_*, x_*) = \mathrm{Val}^*(t_*, x_*).$$

If Isaacs condition is not valid

All statements remain valid if we substitute strategies of the player U with counterstrategies of the player U.

Counterstrategy

The function $U : [t_0, \vartheta_0] \times \mathbb{R}^n \times Q \to P$ is called *counterstrategy*. It is a function of t, x and v.

Step-by-step motion

Let (t_*, x_*) be an initial position, $\Delta = \{t_* = t_0 < t_1 \leq \ldots \leq t_m = \vartheta_0\}$. *Step-by-step motion* is a solution of the equations

$$x[t] = x[\tau_{i-1}] + \int_{\tau_{i-1}}^{t} f(\xi, x[\xi], U(\tau_{i-1}, x[\tau_{i-1}], v[\xi]), v[\xi])d\xi,$$

$$t \in [\tau_{i-1}, \tau_i], \; x[\tau_0] = x_*.$$

Condition

Further we consider only Problems of Degree!

For simplicity let us assume that payoff is $\sigma(x(\vartheta_0))$.

Hamiltonian

$$H(t,x,s) \triangleq \max_{v\in Q}\min_{u\in P}\langle s, f(t,x,u,v)\rangle.$$

Isaacs-Bellman Equation

Equation:

$$\frac{\partial\varphi(t,x)}{\partial t} + H\left(t, x, \frac{\partial\varphi(t,x)}{\partial x}\right) = 0;$$

Boundary condition:

$$\varphi(\vartheta_0, x) = \sigma(x).$$

Minimax Solution

Function φ is a *minimax solution* if for all $(t,x) \in (t_0,\vartheta_0) \times \mathbb{R}^n$ the following inequalities hold:

$$a + H(t,x,s) \leq 0 \ \forall (a,s) \in D^-_{\mathrm{D}}\varphi(t,x);$$

$$a + H(t,x,s) \geq 0 \ \forall (a,s) \in D^+_{\mathrm{D}}\varphi(t,x);$$

Dini Differentials

Dini Subdifferential

$$D^-_{\mathrm{D}}\varphi(t,x) \triangleq \Big\{(a,s) \subset \mathbb{R}\times\mathbb{R}^n : \forall(\tau,g)\in\mathbb{R}\times\mathbb{R}^n$$
$$a\tau + \langle s, g\rangle \leq \liminf_{\alpha\to 0}\frac{\varphi(t+\alpha\tau, x+\alpha g)-\varphi(t,x)}{\alpha}\Big\}.$$

Dini Superdifferential

$$D^+_{\mathrm{D}}\varphi(t,x) \triangleq \Big\{(a,s) \subset \mathbb{R}\times\mathbb{R}^n : \forall(\tau,g)\in\mathbb{R}\times\mathbb{R}^n$$
$$a\tau + \langle s, g\rangle \geq \limsup_{\alpha\to 0}\frac{\varphi(t+\alpha\tau, x+\alpha g)-\varphi(t,x)}{\alpha}\Big\}.$$

Property

If $D^-_{\mathrm{D}}\varphi(t,x)\neq\varnothing$ *and* $D^+_{\mathrm{D}}\varphi(t,x)\neq\varnothing$ *simultaneously, then* $(t,x)\in J$ *and*

$$D^-_{\mathrm{D}}\varphi(t,x)=D^+_{\mathrm{D}}\varphi(t,x)=\{(\partial\varphi(t,x)/\partial t,\nabla\varphi(t,x))\}.$$

Here

- $(\partial\varphi(t,x)/\partial t,\nabla\varphi(t,x))$ is total derivative;
- J denotes the set of points x at which function φ is differentiable. By the Rademacher's theorem measure $[t_0,\vartheta_0]\times\mathbb{R}^n\setminus J$ is 0.

Connection with Clarke Subdifferential

Clarke subdifferential

$$\partial_{\mathrm{Cl}}\varphi(t,x) = \mathrm{co}\{(a,s) : \exists\{t_i,x_i\}_{i=1}^{\infty} \subset J : \\ a = \lim_{i\to\infty} \partial\varphi(t_i,x_i)/\partial t,\ s = \lim_{i\to\infty} \nabla\varphi(t_i,x_i)\}.$$

Inclusions

$$D_{\mathrm{D}}^{-}\varphi(t,x) \subset \partial_{\mathrm{Cl}}\varphi(t,x),\ D_{\mathrm{D}}^{+}\varphi(t,x) \subset \partial_{\mathrm{Cl}}\varphi(t,x).$$

Inverse Problem

Let $\varphi : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be local Lipschitz continuous function such that $\varphi(\vartheta_0, \cdot)$ satisfies the sublinear growth condition.

Design finitely dimensional compacts P and Q, dynamic function f and payoff function σ such that function $\varphi(\cdot, \cdot)$ is a value of differential game

$$\dot{x} = f(t, x, u, v),\ t \in [t_0, \vartheta_0],\ x \in \mathbb{R}, u \in P,\ v \in Q$$

with payoff functional $\sigma(x(\vartheta_0))$.

Conditions

Conditions on f

F1. f and g are continuous;

F2. f and g are locally Lipschitz continuous with respect to x;

F3. for all $t \in [t_0, \vartheta_0]$, $x \in \mathbb{R}^n$, $u \in P$, $v \in Q$

$$\|f(t,x,u,v)\|, |g(t,x,u,v)| \leq \Lambda_f(1 + \|x\|).$$

Conditions on σ

Σ1. σ is locally Lipschitz continuous;

Σ2. for all $x \in \mathbb{R}^n$

$$|\sigma(x)| \leq \Lambda_\sigma(1 + \|x\|).$$

Properties of Hamiltionian

H1. (sublinear growth condition) for all $(t, x, s) \in [t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$

$$|H(t, x, s)| \leq \Lambda_f \|s\|(1 + \|x\|);$$

H2. for every bounded region $A \subset \mathbb{R}^n$ there exist function $\omega_A \in \Omega$ and constant L_A such that for all $(t', x', s'), (t'', x'', s'') \in [t_0, \vartheta_0] \times A \times \mathbb{R}^n$ the following inequality holds:

$$\|H(t', x', s') - H(t'', x'', s'')\| \leq$$
$$\leq \omega(t' - t'') + L_A \|x' - x''\| +$$
$$+ \Lambda_f (1 + \inf\{\|x'\|, \|x''\|\}) \|s_1 - s_2\|;$$

H3. H is positively homogeneous with respect to the third variable: if $\alpha \geq 0$ then

$$H(t, x, \alpha s) = \alpha H(t, x, s).$$

Algorithm

1. Design a set $\mathbb{E} \subset [t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$ and function $h : \mathbb{E} \to \mathbb{R}$ in accordance with the function φ.
2. If the set $\mathbb{E}$ and functions h and φ satisfy some conditions, function φ is a value of some differential game.
3. Extend h to the whole space $[t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$.
4. Design control spaces P, Q and a dynamical function f in accordance with the extension of h.

Steps 3 and 4 can be realize in general way or with the help of some heuristics.

The set $\mathbb{E}$

$\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2$;
$\mathbb{E}_i = \{(t, x, s) : (t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n, \quad s \in E_i(t, x)\} \quad i = 1, 2.$
Set-valued maps $E_1(t, x)$ and $E_2(t, x)$ are defined below.

Points of Differentiability

Let $(t, x) \in J$. Put

$$E_1(t, x) \triangleq \{\nabla\varphi(t, x)\};$$

$$h(t, x, \nabla\varphi(t, x)) \triangleq -\frac{\partial\varphi(t, x)}{\partial t}.$$

Condition (E1)

For any position $(t_*, x_*) \notin J$ and any sequences $\{(t_i', x_i')\}_{i=1}^{\infty}$, $\{(t_i'', x_i'')\}_{i=1}^{\infty} \subset J$ such that $(t_i', x_i') \to (t_*, x_*)$, $i \to \infty$, $(t_i'', x_i'') \to (t_*, x_*)$, $i \to \infty$, the following implication holds:

$$\Big(\lim_{i\to\infty} \nabla\varphi(t_i', x_i') = \lim_{i\to\infty} \nabla\varphi(t_i'', x_i'')\Big) \Rightarrow$$
$$\Big(\lim_{i\to\infty} h(t_i', x_i', \nabla\varphi(t_i', x_i')) = \lim_{i\to\infty} h(t_i'', x_i'', \nabla\varphi(t_i'', x_i''))\Big).$$

Limit Directions

Let $(t,x)\notin J$. Put

$$E_1(t,x)\triangleq\{s\in\mathbb{R}^n:\exists\{(t_i,x_i)\}\subset J: \lim_{i\to\infty}(t_i,x_i)=(t,x)\ \&\ \lim_{i\to\infty}\nabla\varphi(t_i,x_i)=s\}.$$

$E_1(t,x)$ is nonempty and bounded.

Hamiltonian in limit directions

$$h(t,x,s)\triangleq\lim_{i\to\infty}h(t_i,x_i,\nabla\varphi(t_i,x_i))$$
$$\forall\{(t_i,x_i)\}\subset J: \lim_{i\to\infty}(t_i,x_i)=(t,x)\ \&\ s=\lim_{i\to\infty}\nabla\varphi(t_i,x_i).$$

Property

$$\partial_{\mathrm{Cl}}\varphi(t,x)=\mathrm{co}\{(-h(t,x,s),s):s\in E_1(t,x)\}.$$

Designation

$$CJ^- \triangleq \{(t,x) \in (t_0,\vartheta_0) \times \mathbb{R}^n \setminus J : D_{\mathrm{D}}^-\varphi((t,x)) \neq \varnothing\};$$
$$CJ^+ \triangleq \{(t,x) \in (t_0,\vartheta_0) \times \mathbb{R}^n \setminus J : D_{\mathrm{D}}^+\varphi((t,x)) \neq \varnothing\}.$$

Property: $CJ^- \cap CJ^+ = \varnothing$.

- If $(t,x) \in CJ^-$,

  $E_2(t,x) \triangleq \{s \in \mathbb{R}^n : \exists a \in \mathbb{R} : (a,s) \in D_{\mathrm{D}}^-\varphi((t,x))\} \setminus E_1(t,x)$;
- if $(t,x) \in CJ^+$,

  $E_2(t,x) \triangleq \{s \in \mathbb{R}^n : \exists a \in \mathbb{R} : (a,s) \in D_{\mathrm{D}}^+\varphi((t,x))\} \setminus E_1(t,x)$;
- if $(t,x) \in ([t_0,\vartheta_0] \times \mathbb{R}^n) \setminus (CJ^- \cup CJ^+)$

  $E_2(t,x) \triangleq \varnothing$.

Designation

Multivalued Maps

$E(t,x) \triangleq E_1(t,x) \cup E_2(t,x).$

$E^{\natural}(t,x) \triangleq \{\|s\|^{-1}s : s \in E(t,x) \setminus \{0\}\}.$

Subsets of $[t_0,\vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$

$$\mathbb{E}_1 \triangleq \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E_1(t,x)\},$$

$$\mathbb{E}_2 \triangleq \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E_2(t,x)\},$$

$$\mathbb{E} \triangleq \mathbb{E}_1 \cup \mathbb{E}_2 = \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E(t,x)\}.$$

$$\mathbb{E}^{\natural} \triangleq \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E^{\natural}(t,x)\}.$$

Main Result

Let $\varphi : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be local lipschitzian function such that $\varphi(\vartheta_0, \cdot)$ satisfies sublinear growth condition.
Function φ is a value of some differential game with terminal payoff if and only if the function h defined on $\mathbb{E}_1$ is extendable on the set $\mathbb{E}_2$ such that conditions (E1)–(E4) hold. (Conditions (E2)–(E4) are defined below.)

Condition (E2)

- If $(t,x) \in CJ^-$ then for any $s_1, \dots s_{n+2} \in E_1(t,s)$ $\lambda_1, \dots, \lambda_{n+2} \in [0,1]$ such that
  $\sum \lambda_k = 1$, $(-\sum \lambda_k h(t,x,s_k), \sum \lambda_k s_k) \in D^-\varphi(t,x)$
  the following inequality holds:
  $$h\left(t, x, \sum_{k=1}^{n+2} \lambda_k s_k\right) \leq \sum_{k=1}^{n+2} \lambda_k h(t,x,s_k);$$

- If $(t,x) \in CJ^+$ then for any $s_1, \dots s_{n+2} \in E_1(t,s)$ $\lambda_1, \dots, \lambda_{n+2} \in [0,1]$ such that
  $\sum \lambda_k = 1$, $(-\sum \lambda_k h(t,x,s_k), \sum \lambda_k s_k) \in D^+\varphi(t,x)$
  the following inequality holds:
  $$h\left(t, x, \sum_{k=1}^{n+2} \lambda_k s_k\right) \geq \sum_{k=1}^{n+2} \lambda_k h(t,x,s_k);$$

Condition (E3)

- if $0 \in E(t,x)$, then $h(t,x,0) = 0$;
- if $s_1 \in E(t,x)$ and $s_2 \in E(t,x)$ are codirectional (i.e. $\langle s_1, s_2 \rangle = \|s_1\| \cdot \|s_2\|$), then
$$\|s_2\| h(t,x,s_1) = \|s_1\| h(t,x,s_2).$$

Function $h^\natural : \mathbb{E}^\natural \to \mathbb{R}$

$\forall (t,x) \in [t_0, \vartheta_0] \times \mathbb{R}^n \ \forall s \in E(t,x) \setminus \{0\}$
$$h^\natural(t,x,\|s\|^{-1}s) \triangleq \|s\|^{-1} h(t,x,s).$$

Condition (E4)

Sublinear growth condition: there exists $\Gamma > 0$ such that for any $(t, x, s) \in \mathbb{E}^\natural$ the following inequality is fulfilled

$$h^\natural(t, x, s) \leq \Gamma(1 + \|x\|).$$

Difference estimate:

For every bounded region $A \subset \mathbb{R}^n$ there exist $L_A > 0$ and modulus of continuity ω_A such that for any $(t', x', s'), (t'', x'', s'') \in \mathbb{E}^\natural \cap [t_0, \vartheta_0] \times A \times \mathbb{R}^n$ the following inequality is fulfilled

$$|h^\natural(t', x', s') - h^\natural(t'', x'', s'')| \leq \omega_A(t' - t'') + \\ + L_A\|x' - x''\| + \Gamma(1 + \inf\{\|x'\|, \|x''\|\})\|s' - s''\|.$$

A method of extension

Suppose that h as function defined on $\mathbb{E}_1$ satisfies condition $(E1)$. Suppose also that the extension of h on $\mathbb{E}_2$ given by the following rule is well defined: for all $(t,x) \in CJ^- \cup CJ^+$, $s \in E_2(t,x)$, $s_1, \ldots, s_{n+2} \in E_1(t,x)$, $\lambda_1, \ldots, \lambda_{n+2} \in [0,1]$ such that $\sum \lambda_i = 1$ $\sum \lambda_i s_i = s$

$$h(t,x,s) \triangleq \sum_{i=1}^{n+2} \lambda_i h(t,x,s_i).$$

If function $h : \mathbb{E} \to \mathbb{R}$ satisfies conditions (E3) and (E4), then φ is a value of some differential game with terminal payoff functional.

Scheme of Proof

Step 0

Define payoff functional by formula $\sigma(\cdot) \triangleq \varphi(\vartheta_0, \cdot)$

Step 1

- Extend function $h^\natural$ defined on $\mathbb{E}^\natural$ to the set $[t_0, \vartheta_0] \times \mathbb{R}^n \times S^{(n-1)}$. ($S^{(k)}$ is k-dimensional sphere). Denote this extension by h^*.
- Design the positively homogeneous function $H : [t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is an extension of h^*.

Step 2

Design finitely dimensional compacts P, Q and function f in accordance with H.

Positive Example

Let $n = 2$, $t_0 = 0$, $\vartheta_0 = 1$.

$$\varphi_1(t, x_1, x_2) = t + |x_1| - |x_2|.$$

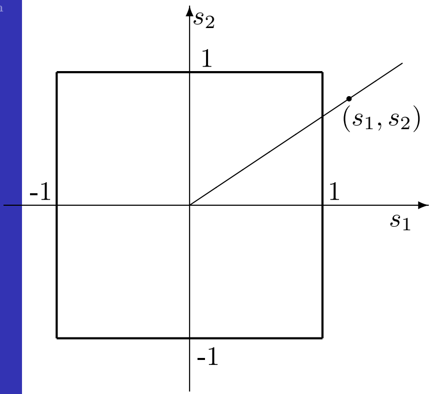
Function h

For $x_1, x_2 \neq 0$	$h(t, x_1, x_2; \mathrm{sgn} x_1, \mathrm{sgn} x_2) = -1.$
For $x_1 = 0,\ x_2 \neq 0$	$h(t, 0, x_2; \pm 1, \mathrm{sgn} x_2) = -1.$
For $x_1 \neq 0,\ x_2 = 0$	$h(t, x_1, 0; \mathrm{sgn} x_1, \pm 1) = -1.$
For $x_1 = x_2 = 0$	$h(t, 0, 0; \pm 1, \pm 1) = -1.$

Sets

$$J = \{(t, x_1, x_2) : x_1 x_2 \neq 0\}.$$
$$CJ^- = \{(t, 0, x_2) : x_2 \neq 0\},$$
$$CJ^+ = \{(t, x_1, 0) : x_1 \neq 0\}.$$

Specific Method of Reconstruction



H on square

For (s_1, s_2) such that $\max\{|s_1|, |s_2|\} = -1$ $H(s_1, s_2) \triangleq -1$.

$$H(s_1, s_2) = \\ = \min\{-|s_1|, -|s_2|\}.$$

Control system

$$\begin{cases} \dot{x}_1 = & u_0 u_1, \\ \dot{x}_2 = & (1 - u_0) u_2. \end{cases}$$

$u_0 \in \{0, 1\}$, $u_1, u_2 \in [-1, 1]$.

Negative Example

Let $n = 2$, $t_0 = 0$, $\vartheta_0 = 1$.

$$\varphi_2(t, x_1, x_2) = t(|x_1| - |x_2|).$$

Analysis

$J = \{(t, x_1, x_2) : t \in (0, 1), x_1 x_2 \neq 0\}$.

For $(t, x) \in J$ $\quad E(t, x) = \{(t \cdot \mathrm{sgn} x_1, t \cdot \mathrm{sgn} x_2)\}$.

$$h(t, x_1, x_2; t \cdot \mathrm{sgn} x_1, t \cdot \mathrm{sgn} x_2) = |x_1| - |x_2|.$$

$$\mathbb{E}_0 \triangleq \{(t, x_1, x_2; t\mathrm{sgn} x_1, t\mathrm{sgn} x_2) : (t, x_1, x_2) \in J\}.$$

$$\mathbb{E}_0^\natural \triangleq \{(t, x_1, x_2; \mathrm{sgn} x_1/\sqrt{2}, \mathrm{sgn} x_2/\sqrt{2}) : (t, x_1, x_2) \in J\}.$$

If $(t, x_1, x_2) \in J$

$$h^\natural(t, x_1, x_2; \mathrm{sgn} x_1/\sqrt{2}, t\mathrm{sgn} x_2/\sqrt{2}) = \frac{|x_1| - |x_2|}{\sqrt{2}t}.$$

Bibliography

- *Krasovskii N.N., Subbotin A.I.* Game-Theoretical Control Problems, New York: Springer, 1988;
- *Subbotin A.I.* Generalized solutions of first-order PDEs. The dynamical perspective, Systems & Control: Foundations & Applications, Birkhauser, Boston, Ins., Boston MA, 1995
- *Bardi M, Capuzzo-Dolcetta I.* Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia, Boston. Systems & Control: Foundations & Applications. Birkhauser Boston, Inc. 1997, xviii+570 pp.
- *Demyanov V.F., Rubinov A.M.* Foundations of Nonsmooth Analysis, and Quasidifferential Calculus, Optimization and Operation Research, v. 23, Nauka, Moscow, 1990, 431pp.
- *McShane E. J.* Extension of range of function // Bull.Amer.Math.Soc. 1934. V. 40. пїЅ12, Pp 837–842.
- *Evans L.C., Souganidis P.E.* Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs Equations // Indiana University Mathematical Journal, 1984, Vol. 33, N 5, Pp. 773–797.