Characterization of feedback Nash equilibrium for differential games

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Outline

Nonzero-sum differential game

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

Here u and v are controls of the player I and the player II respectively.

- The player I wants to maximize $\sigma_1(x(\vartheta_0))$.
- The player II wants to maximize $\sigma_2(x(\vartheta_0))$.

Conditions

- The sets P and Q are compacts.
- The functions f, σ_1 and σ_2 are continuous;
- The function f is locally Lipschitz continuous with respect to the phase variable
- The function f satisfies the sublinear growth condition with respect to x
- Isaacs condition holds.

N.N. Krasovskii discontinuous feedback formalization

- The strategy of the player I: $U = (u(t, x, \varepsilon_1), \beta_1(\varepsilon_1))$.
- The strategy of the player II: $V = (v(t, x, \varepsilon_2), \beta_2(\varepsilon_2))$.

ε_1 and ε_2 are precision parameters of the players.

Control design

- The player I chooses a precision parameter ε_1 and a partition $\Delta_1 = \{t'_i\}_{i=0}^m$.
  $u(t) = u(t'_i, x[t'_i], \varepsilon_1)$, $t \in [t'_i, t'_{i+1})$.
- The player II chooses a precision parameter ε_2 and a partition $\Delta_2 = \{t''_i\}_{i=0}^r$.
  $v(t) = v(t''_i, x[t''_i], \varepsilon_2)$, $t \in [t''_i, t''_{i+1})$.

Fineness(Δ_i*)*$\leq \beta_i(\varepsilon_i)$, $i = 1, 2$.

Bundles of motions

N.N. Krasovskii [Game-Theoretical Control Problems],
A.F. Kleimenov [Non zero-sum differential games]

- Step-by-step motion.
- Consistent step-by-step motion $\varepsilon_1 = \varepsilon_2$.
- Set of constructive motions $X(t_*, x_*; U, V)$.
- Set of consistent constructive motions $X^c(t_*, x_*; U, V)$.

Any limit of a sequence of (consistent) step-by-step motions is called (consistent) constructive motions.

Nash equilibrium

The pair of strategies U^N and V^N is said to be *Nash equilibrium solution* at the position (t_*, x_*), if for all strategies U and V the following inequalities hold:

$$\max\{\sigma_1(x[\vartheta_0]) : x[\cdot] \in X(t_*, x_*; U, V^N)\} \leq \\ \leq \min\{\sigma_1(x^c[\vartheta_0]) : x^c[\cdot] \in X^c(t_*, x_*; U^N, V^N)\}.$$

$$\max\{\sigma_2(x[\vartheta_0]) : x[\cdot] \in X(t_*, x_*; U^N, V)\} \leq \\ \leq \min\{\sigma_2(x^c[\vartheta_0]) : x^c[\cdot] \in X^c(t_*, x_*; U^N, V^N)\}.$$

Nash equilibrium

Nash value of the game is

$$\mathcal{N}(t_*, x_*) = \{(\sigma_1(x[\vartheta_0]), \sigma_2(x[\vartheta_0])) : \quad x[\cdot] \in X^c(t_*, x_*; U^N, V^N).$$

Properties:

- The set $\mathcal{N}(t_*, x_*)$ is nonempty.
- In general $\mathcal{N}(t_*, x_*)$ contains infinitely many couples.

Auxiliary zero-sum games

- *Game* Γ_1*:* The player I wants to maximize $\sigma_1(x(\vartheta_0))$, the purpose of the player II is opposite. Denote the value of this game by $\omega_1 : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$.
- *Game* Γ_2*:* The player II wants to maximize $\sigma_2(x(\vartheta_0))$, the purpose of the player I is opposite. Denote the value of this game by $\omega_2 : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$.

Auxiliary differential inclusion

$$\dot{x} \in \mathcal{F}(t,x) \triangleq \mathrm{co}\{f(t,x,u,v) : u \in P, v \in Q\}$$

By $\mathrm{Sol}(t_*, x_*)$ denote the set of its solution with initial data $x(t_*) = x_*$.

Multivalued map

Statement

Let a multivalued map $\mathcal{T} : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^2)$ satisfy the following conditions

(N1) $\mathcal{T}(\vartheta_0, x) = \{(\sigma_1(x), \sigma_2(x))\}$ for all $x \in \mathbb{R}^n$;

(N2) $\mathcal{T}(t, x) \subset [\omega_1(t, x), \infty) \times [\omega_2(t, x), \infty)$ for all $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$;

(N3) for all $(t_*, x_*) \in [t_0, \vartheta_0] \times \mathbb{R}^n$, $(J_1, J_2) \in \mathcal{T}(t_*, x_*)$ there exists $y(\cdot) \in \mathrm{Sol}(t_*, x_*)$ such that

$$(J_1, J_2) \in \mathcal{T}(t, y(t)) \quad t \in [t_*, \vartheta_0].$$

Then $\mathcal{T}(t, x) \subset \mathcal{N}(t, x)$ for all $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$.

The multivalued map $\mathcal{N}$ satisfies the conditions (N1)–(N3).

Semicontinuous maps

Definition. The multivalued map $\mathcal{T}$ is upper semicontinuous by inclusion if its graph is closed.

Property. The multivalued map $\mathcal{N}$ is upper semicontinuous by inclusion.

Characterization in the terms of weak invariance

Theorem 1

Let $\mathcal{T} : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^2)$ be upper semicontinuous by inclusion.

Condition (N3) is fulfilled if and only if for any $(t_*, x_*) \in [t_0, \vartheta_0] \times \mathbb{R}^n$ and $(J_1, J_2) \in \mathcal{T}(t_*, x_*)$ there exist $\theta > t_*$ and $y(\cdot) \in \mathrm{Sol}(t_*, x_*)$ such that

$$(J_1, J_2) \in \mathcal{T}(t, y(t)), \quad t \in [t_*, \theta].$$

Derivative of multivalued map

Distance:

Let $A \subset \mathbb{R}^2$,

$$\mathrm{dist}[(J_1, J_2), A] \triangleq \inf \{|\zeta_1 - J_1| + |\zeta_2 - J_2| : (\zeta_1, \zeta_2) \in A\}.$$

Directional derivative of the multivalued map:

$$\mathrm{D}_H \mathcal{T}(t, x; (J_1, J_2), w) \triangleq \liminf_{\delta \downarrow 0, w' \to w} \frac{\mathrm{dist}[(J_1, J_2), \mathcal{T}(t + \delta, x + \delta w')]}{\delta}.$$

Characterization in the terms of nonsmooth analysis

Theorem 2

Let $\mathcal{T} : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^2)$ be upper semicontinuous by inclusion.

Condition (N3) is valid if and only if for any $(t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n$

$$\sup_{(J_1,J_2)\in\mathcal{T}(t,x)} \inf_{w\in\mathcal{F}(t,x)} \mathrm{D}_H\mathcal{T}(t, x; (J_1, J_2), w) = 0.$$

Sufficient condition

Let $(c_1, c_2) : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}^2$ be a continuous function.

Is $(c_1(t,x), c_2(t,x))$ *a Nash equilibrium payoff?*

Definitions

Modulus derivative:

$$\mathrm{d}_{abs}(c_1, c_2)(t, x; w) \triangleq \liminf_{\delta \downarrow 0, w' \to w} \frac{|c_1(t+\delta, x+\delta w') - c_1(t, x)| + |c_2(t+\delta, x+\delta w') - c_2(t, x)|}{\delta}.$$

Auxiliary Hamiltonians:

$$H_1(t, x, s) \triangleq \max_{u \in P} \min_{v \in Q} \langle s, f(t, x, u, v) \rangle,$$

$$H_2(t, x, s) \triangleq \min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v) \rangle.$$

Sufficient condition in the infinitesimal form

Theorem 3

Suppose that

- $(c_1(\vartheta_0,\cdot),c_2(\vartheta_0,\cdot))=(\sigma_1(\cdot),\sigma_2(\cdot))$;
- The functions c_i are the upper solution of the equations
$$\frac{\partial c_i}{\partial t}+H_i(t,x,\nabla c_i)=0,\quad i=1,2,$$
- for all $(t,x)\in[t_0,\vartheta_0]\times\mathbb{R}^n$
$$\inf_{w\in\mathcal{F}(t,x)}\mathrm{d}_{abs}(c_1,c_2)(t,x;w)=0.$$

Then $(c_1(t,x),c_2(t,x))$ *a Nash equilibrium payoff* for all $(t,x)\in[t_0,\vartheta_0]\times\mathbb{R}^n$.

Connection with the system of HJ PDEs

$$\mathcal{H}_i(t,x,s_1,s_2) \triangleq \langle s_i, f(t,x,u^n,v^n)\rangle, \quad i=1,2.$$

$$\max_{u\in P}\langle s_1, f(t,x,u,v^n)\rangle = \langle s_1, f(t,x,u^n,v^n)\rangle,$$

$$\max_{v\in Q}\langle s_2, f(t,x,u^n,v)\rangle = \langle s_2, f(t,x,u^n,v^n)\rangle.$$

Connection with the system of HJ PDEs

Statement

If the function (φ_1, φ_2) is a solution of the system

$$\frac{\partial \varphi_i}{\partial t} + \mathcal{H}_i(t, x, \nabla\varphi_1, \nabla\varphi_2) = 0, \quad i = 1, 2.$$

Then (φ_1, φ_2) satisfies the conditions of Theorem 3.

Example. The game

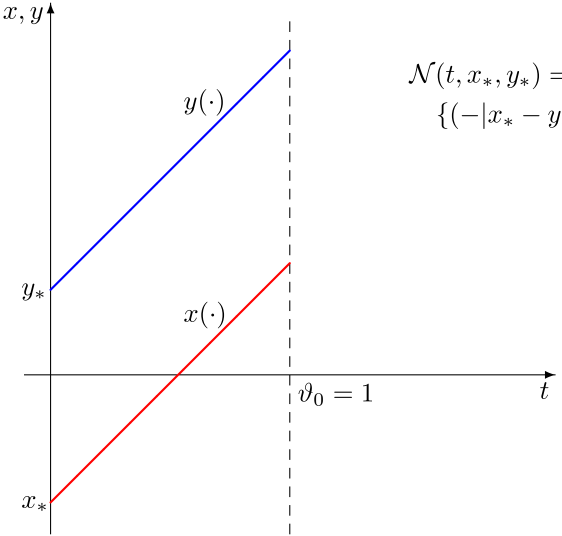
$$\begin{cases} \dot{x} & = & u \\ \dot{y} & = & v \end{cases}$$

$t \in [0, 1]$, $u, v \in [-1, 1]$.

The player I wants to maximize $\sigma_1(x, y) \triangleq -|x - y|$.

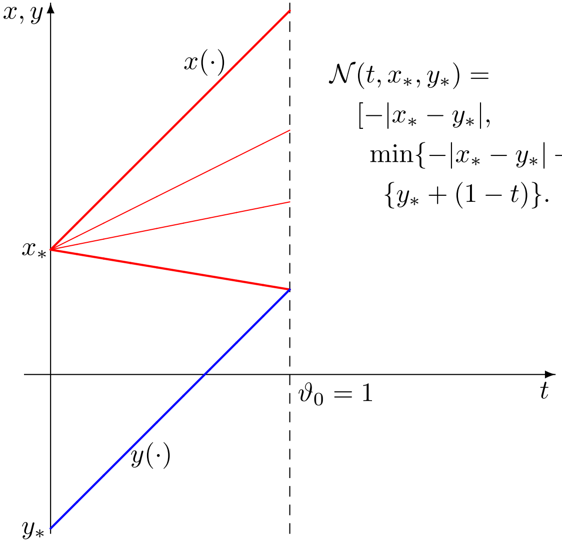
The player II wants to maximize $\sigma_2(x, y) \triangleq y$.

Case $y_* > x_*$



$$\mathcal{N}(t, x_*, y_*) = \{(-|x_* - y_*|, y_* + (1 - t))\}.$$

Case $x_* \geq y_*$



$$\mathcal{N}(t, x_*, y_*) = [-|x_* - y_*|, \min\{-|x_* - y_*| + 2(1-t), 0\}] \times \{y_* + (1-t)\}.$$

System of Hamilton-Jacobi equations

$$\begin{cases} \frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_1}{\partial x}u^n(t,x,y) + \frac{\partial\varphi_1}{\partial y}v^n(t,x,y) & = 0 \\ \frac{\partial\varphi_2}{\partial t} + \frac{\partial\varphi_2}{\partial x}u^n(t,x,y) + \frac{\partial\varphi_2}{\partial y}v^n(t,x,y) & = 0. \end{cases}$$

Boundary condition: $\varphi_1(1,x,y) = -|x-y|$, $\varphi_2(1,x,y) = y$.

Here $u^n(t,x,y)$ and $v^n(t,x,y)$ satisfy the conditions

$$\frac{\partial\varphi_1}{\partial x}u^n(t,x,y) = \max_{u\in P}\left[\frac{\partial\varphi_1}{\partial x}u\right], \quad \frac{\partial\varphi_1}{\partial x}v^n(t,x,y) = \max_{u\in P}\left[\frac{\partial\varphi_1}{\partial x}v\right].$$

There is no classical solution.

System of Hamilton-Jacobi equations

Viscosity solution

$$\varphi_1(t,x,y) = \begin{cases} x-y, & x \leq y, \\ -x+y+2(1-t), & x > y, -x+y+2(1-t) < 0, \\ 0, & x > y, -x+y+2(1-t) \geq 0 \end{cases}$$

$$\varphi_2(t,x,y) = y + (1-t).$$

The couple $(\varphi_1(t,x_*,y_*), \varphi_2(t,x_*,y_*))$ *is maximum Nash equilibrium payoff at the position* (t,x_*,y_*).

Functions providing Nash equilibria

$$c_1^\gamma(t, x_*, y_*) = \begin{cases} -|x-y|, & y \geq x; \\ \min\{-|x-y| + \gamma(1-t); 0\}, & y < x \end{cases}$$

$$c_2(t, x, y) = y + (1-t).$$

The function (c_1^γ, c_2) *satisfies the conditions of the Theorem 3 for* $\gamma \in [0, 2]$*.*

Reference

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